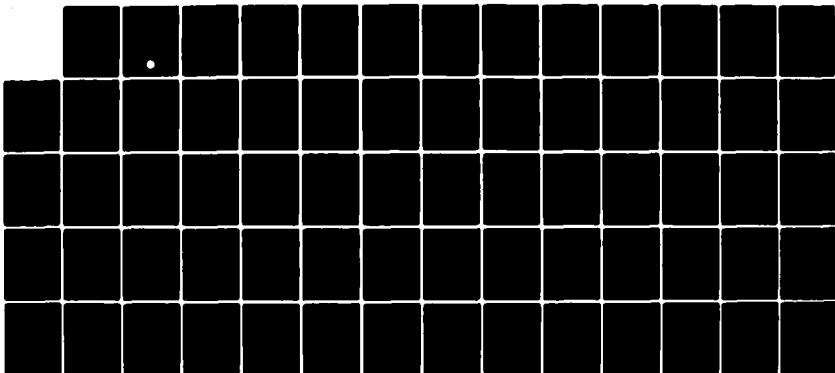AD-A130 453   OCCUPANCY MODELS BELL-TYPE POLYNOMIALS AND NUMBERS AND   1/ 1
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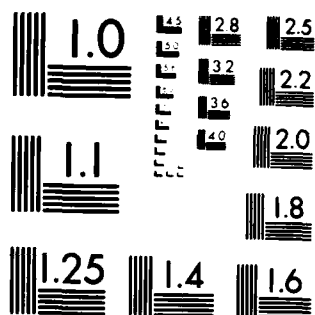UNCLASSIFIED   F/G 12/1   NL

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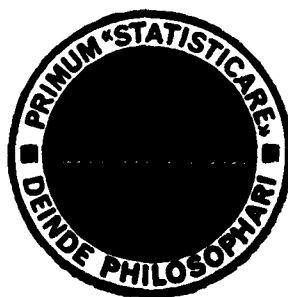
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OCCUPANCY MODELS, BELL-TYPE POLYNOMIALS
AND NUMBERS AND APPLICATIONS TO PROBABILITY

Final Technical Report

by

T. Cacoullos

June 1983

US ARMY RESEARCH AND STANDARDIZATION GROUP (EUROPE)

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Multipartitional extensions and modifications of Bell polynomials as well as related analogues of Stirling and C-numbers are shown to be a natural and strong tool in the study of compound distributions, fluctuation theory and combinatorial distributions. Combinatorial and occupancy model aspects of these numbers are discussed. Characterizations of compound distributions are also given.

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OCCUPANCY MODELS, BELL-TYPE POLYNOMIALS AND NUMBERS
AND APPLICATIONS TO PROBABILITY

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Abstract

Multipartitional extensions of Bell (unipartitional) polynomials are shown to be a natural and strong tool in the study of multivariate compound discrete distributions through their generating functions. Modifications of exponential polynomials simplify proofs in fluctuation theory, whereas asymptotic properties of such polynomials are used to establish the asymptotic normality of a wide class of combinatorial distributions, including Stirling and C-numbers. Extensions of these numbers, the non-central Stirling numbers and the mutli-parameter Stirling and C-numbers are studied in conjunction with distributional, estimation and characterization problems related to compound distributions. Combinatorial and occupancy-model aspects are also discussed. Diagnostic tests in data analysis are pointed out.

Keywords: Multipartitional Bell polynomials, non-central Stirling numbers, multiparameter Stirling and C-numbers, combinatorial distributions, fluctuation theory, compound distributions, characterizations, regression function.

OCCUPANCY MODELS, BELL-TYPE POLYNOMIALS AND NUMBERS
AND THEIR APPLICATION IN PROBABILITY THEORY

1. GENERAL BACKGROUND OF RESEARCH AREA

The development of classical probability theory has its origins in games of chance and is primarily of the discrete type. Even today, it is customary to introduce and describe many discrete probability models in terms of urn or occupancy models. The models themselves and their ramifications involve concepts and methods of combinatorial analysis. It is not, then, surprising that combinatorics, in one form or another, has always been in the foreground of probabilistic and statistical-inferential arguments since the times of Pascal and Fermat.

However, in spite of an abundance of probability-related combinatorial results in the vast mathematical literature over a long period of two or three centuries, many of the results are "randomly" and sparsely scattered in so many books and professional journals so that they either remain unknown to the majority of professionals or keep being rediscovered all the time. The gap was widening also due to the fact that certain closely related areas of mathematics, such as the calculus of finite differences, were neglected in modern curricula. It is only recently that attempts have been made to tie up such results in a systematic manner. For these reasons, monographs such as Jordan (1950), David and Barton (1962) and Johnson and Kotz (1977) signify a good move towards a unified approach of the methodology in the wide-scope area of combinatorial probability.

This research concerns: occupancy or urn models, Bell or partition polynomials and certain kinds of numbers (integers) relating to Bell polynomials such as Bell numbers. One can arrive at these numbers from different directions, as will become clear in the sequel; nevertheless we cover all such numbers, which are also connected to occupancy models, under the name "Bell-type" numbers.

Occupancy models and distributions are well understood. Urn models constitute the usual, though not the only way to introduce most discrete distributions. The recent monograph by Johnson and Kotz (1977) provides a comprehensive exposition of this approach.

Partition polynomials, also called Bell polynomials by Riordan (1958, 1968) are connected with the derivatives of a composite function and provide a powerful tool in the treatment of combinatorial and probabilistic problems. Let

$$A(t) = f(g(t))$$

and define $D_t = d/dt$, $D_u = d/du$, (1)

$$A_n = D_t^n A(t), \quad f_n = D_u^n f(u) \quad \text{with} \quad u = g(t), \quad D_t^n g(t) = g_n.$$

Then

$$A_n = Y_n(f; g_1, \ldots, g_n) \tag{2}$$

where Y_n is a partition polynomial defined by

$$Y_n(f; g_1, \ldots, g_n) = \sum_{\pi(n)} \frac{n! \, f_k}{k_1! \ldots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \ldots \left(\frac{g_n}{n!}\right)^{k_n}$$

and the summation is over all partitions $\pi(n)$ of n, i.e., all non-negative integers $k_1, \ldots, k_n$ such that

$$k_1 + 2k_1 \ldots + nk_n = n,$$

and

$$k = k_1 + k_2 + \ldots + k_n$$

represents the number of parts in a given partition. The SO called exponential po-
lynomials, $E_n(g_1,\ldots g_n)$, introduced by Bell (1934 a, b,), are a special case of Y_n,
namely, when $f_k = 1$, $k = 1, 2,\ldots$ Thus

$$E_n(g_1,\ldots g_n) = Y_n(1; g_1,\ldots g_n) = e^{-g} D_t^n e^g , \quad g \cong g(t) \tag{3}$$

The exponential generating function (egf) of the sequence $\{Y_n,\ n \geqslant o\}$ $(Y_o=1)$
of Bell polynomials can be written in the form:

$$\exp (uY) = \sum_{n=0}^{\infty} Y_n(f; g_1,\ldots g_n) \frac{u^n}{n!}$$
$$= \exp\left[f(\sum_{k=1}^{\infty} \frac{u^k g_k}{k!}) \right] = \exp\left[f\ G\ (u)\right], \tag{4}$$

where, in the exponential expansions, we set

$$G\ (u) = \exp\left[(ug) - g_o\right]\quad f^k \cong f_k , \quad g^k \cong g_k, \quad Y^n \cong Y_n. \tag{5}$$

It should be mentioned that (4), in conjunction with (5), attests to the
general fact that the algebra associated with egf's is what is known as the Blis-
sard (or symbolic or umbral) calculus; the algebra of ordinary generating functi-
ons is known as the Cauchy algebra.

In the umbral calculus, a sequence a_n may be replaced by the sequence
a^n of powers and when all operations are performed the exponents are changed
back to indices. For example, if A (u), B (u) and C (u) are the egf's of the se-
quences a_k, b_k and c_k , respectively, and

$$C\ (u) = A\ (u) \cdot B\ (u),$$

then

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

so that, in the umbral calculus notation,

$$c_n = (a + b)^n, \quad a^n \cong a_n , \quad b^n \equiv b_n ;$$

the egf's behave like exponential functions:

$$C (u)= \exp (cu)= \exp (a + b)= \exp (au) \exp (bu)= A (u) B (u)$$

The exponential polynomial $E_n(t_1,\ldots t_n)$ is closely related to the generating function (gf), Riordan (1958),

$$C_n(t_1,\ldots t_n) = \sum_{\pi(n)} C (k_1,\ldots k_n) t_1^{k_1}\ldots t_n^{k_n}, \tag{6}$$

of the number $C (k_1,\ldots k_n)$ of permutations of n elements with k_1 unit cycles, k_2 2-cycles, etc. C_n is the <u>cycle indicator</u> of the symmetric group and is expressed in terms of $E_n(t_1,\ldots t_n)$:

$$C_n(t_1,\ldots t_n)=E_n(t_1, t_2, 2! t_3,\ldots,(n-1)! t_n).$$

The polynomials E_n are connected with several enumeration problems. For details,we refer e.g., to Riordan (1958, 1968). We mention here, e.g., that $C_n (t,\ldots t)$ is equal to the gf.of the signless Stirling numbers of the first kind $c(n,k) = (-1)^{k+n} s(n,k)$, where $s(n,k)$ are the Striling numbers of the first kind; $c(n,k)$ is equal to the number of permutations of n elements with k cycles; similarly, $C_n(0, t,\ldots,t)$ is the gf of the number of permutations with k cycles no one of which is a unitary cycle. Moreover, the polynomials $E_n(t_1,\ldots t_n)$ themselves,in exactly the same manner as the C_n, are related to ordered cycles of permutations. E_n is the ordered-cycle indicator and is associated with Stirling numbers of the second kind, $S(n,k)$. For example, the polynomials

$$S_n (x) = E_n(x,\ldots x) = \sum_{k=0}^{n} S (n,k) x^k \tag{7}$$

for $x= 1$ give the <u>Bell numbers</u>; $S_n(1)$ equals the number of partitions of n.

Another application of $E_n(g_1,\ldots g_n)$ with $g_k= (s)_k x$, $s> 0$ or an integer, leads to the polynomials

$$C_{n,s}(x)= E_n((s)_1 x ,\ldots (s)_n x) = \sum_{k=0}^{n} C (n, k, s) x^k,$$

where the C-numbers C (n,k,s) were introduced by Cacoullos and Charalambides(1975) and further studied by Charalambides (1974, 1977). These authors showed in a series of papers (see also Cacoullos 1977) that Stirling numbers of the first and second kind and C-numbers, as well as certain generalizations of these, emerge,quite naturally,in a unified treatment, via eqf's, of the minimum variance unbiased estimation problem for left-truncated logarithmic series, Poisson, binomial and negative binomial distributions.

The role played by Stirling numbers in occupancy and distribution problems is well known; also their interpretation in terms of cycles of permutations (Riordan,1958). Analogous combinatorial interpretations,along with their probabilistic counterparts, can be given for the C-numbers and their generalizations. If s is a positive integer the numbers C (n,k,s) are associated with the classical coupons collector's problem. If s is a negative integer then $n! \cdot C(n,k,-s) /k!$ equals the number of ways n> k indistinguishable balls can be distributed into k groups, each of s cells, so that each group contains at least one ball. The generalized C-numbers, C(n,k,s,r),correspond to the situation where each group is to contain at least r balls (Charalambides, 1974).

The above type of numbers appear in other contexts as well. For example, the generalized Stirling numbers of the second kind, in our notation S (n,b,r),coincide with the numbers $S_{(n,r)}^{(b,j)}$ for j=o, in the notation of Sobel et al (1977). They were introduced in the study of the Type 1-Diriclet integral $I_p^{(k)}$ (r,n). This is a multivariate extension of the incomplete beta function, I_p, for multinomial-related probabilities. In fact,

$$S(n,b,r) \frac{b^n}{b!} = I_{1/b}^{(b)}(r,n) \tag{8}$$

and $I_{1/b}^{(b)}(r,n)$ is the pr ability that each cell in a binomial distribution

receives at least r balls. Tables of $S(n,b,r)$ are given in Sobel (op.cit.) for $b=1$ (1) 23, $n=b$(1) 25, $r=1$ (1) $[n/b]$. For $b=1$, the incomplete beta function

$$I_p^{(1)} (r.n) = I_p (r,n-r+1) = \frac{1}{B(r,n+1-r)} \int_0^p x^{r-1} (1-x)^{n-r} dx$$

can be written as

$$I_p^{(1)} (r,n) = p^r \sum_{j=0}^{n-r} \frac{\Gamma(r+j)}{\Gamma(r) j!} q^j \;,$$

whose generalization leads to a Type 2-Dirichlet integral (see Cacoullos and Sobel, 1966).

Apparently, the first systematic attempt to use Bell polynomials in probability has been the recent study, Charalambides (1977) of compound (generalized) discrete distributions. Let N and X_i be independent integer -valued random variables with probability generating functions (pgf) f and g, respectively. Then the pgf, $P(t)$ say, of the compound distribution of

$$S_N = X_1 + X_2 + \ldots + X_N$$

is given by the compound function $P(t) = f(g(t))$ and, in view of (1) and (2) and the elementary result

$$P_n (f,g) = P [S_N = n] = \frac{1}{n!} D_t^n P(t) \Big|_{t=0} \;,$$

we readily obtain the compound probability function in the form

$$P_O (f,g) = f(g(0))$$

$$P_n (f,g) = \frac{1}{n!} Y_n (f; g_1, \ldots, g_n) \tag{9}$$

where $\quad f_n = D_u^n f(u) \big|_{u=g(o)} \;, \quad g_n = D_t^n g(t) \big| t=o.$

From a recurrence relation for Y_n one obtains the basic recurrence

$$P_{n+1} (f,g) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{n!} g_{k+1} f P_{n-k}(f,g) \;, \qquad f^k = f_k .$$

Similar results can be given for the moments of S_N.

INTRODUCTION AND SUMMARY OF RESEARCH

Within the general scope of the research project, the results, are exhibited in the following Sections 2-8. Whenever the results are already published, only main ideas and summaries are given in these sections. Whenever the work is still in the process of publication, it is presented in an appendix.

Section 2 gives some general theory concerning compound (generalized) discrete distributions. It also discusses ad hoc estimation procedures in conjunction with the problem of modeling certain real accident data by using the usual discrete distributions.

Section 3 presents extensions of Bell polynomials appropriate for the treatment of multivariate compound distributions.

Section 4 deals with the asymptotic normality of general combinatorial distributions, including Stirling and C-numbers as special cases.

In Section 5, non-central Stirling numbers of the first and second kind are defined and their applications in convolutions of classical discrete distributions as well as their combinatorial interpretations in occupancy models are discussed. Another extension, the multiparameter Stirling and C-numbers, motivated by the estimation problem for multiply truncated power series distributions is further discussed in Appendix A.

Some modifications of Bell polynomials useful in simplifying proofs in fluctuation theory are given in Section 6.

Finally, Section 7 looks at compounding from the point of view of mixtures of distributions. It turns out that the regression function of the mixing variable on the mixture (compound) variable in conjunction with identifiability results yields characterizations both for discrete and continuous mixtures. Since the regressions in the discrete case are closely related to Bell-type polynomials and numbers, the results are given in Appendix B.

2. MULTIVARIATE DISCRETE MODELS GENERATED BY COMPOUNDING

Multivariate discrete models $(N,\underset{\sim}{Z})$ generated by compounding (generalizing) an integer-valued positive random variable (r.v.) N by a d-variate discrete random vector $\underset{\sim}{\xi} = (X_1,\ldots,X_d)$ are investigated via probability generating functions (p.g.f.) and it is shown that Bell polynomials and related numbers (e.g. Stirling and C-numbers) play an important role, not only in expressing probabilities and moments, but also in explicit representations of the conditional distribution of N given $\underset{\sim}{Z}$. More specifically, the (compound) distribution of Z is determined by the representation

$$\underset{\sim}{Z} = \underset{\sim}{\xi}_1 + \cdots + \underset{\sim}{\xi}_N \tag{1}$$

where $\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2,\ldots$ are independent observations on $\underset{\sim}{\xi}$, which is assumed independent of the generalized r.v. N.

Several applications lead to such models. For example, N may represent the number of car accidents in a given locality during certain time period, X_1 the corresponding injury accidents, X_2 the fatal accidents, X_3 the injuries and X_4 the fatalities. The joint behavior of N and any or all of the X_i requires an $(N,\underset{\sim}{Z})$-model.

The emergence of Bell partition (d=1) and multipartitional (Section 3) polynomials in the study of the probabilistic structure of an $(N,\underset{\sim}{Z})$-model is due to the follwoing basic facts:

1. The p.g.f. G of $(N,\underset{\sim}{Z})$ is given by

$$G(u,\underset{\sim}{v}) = g_1(u \cdot g_2(\underset{\sim}{v})) \tag{2}$$

where g_1 is the p.g.f. of N and g_2 the p.g.f. of $\underset{\sim}{\xi}$; the p.g.f. of the compound distribution of $\underset{\sim}{Z}$ is the compound function $g_1(g_2(\underset{\sim}{v})) = G(1,\underset{\sim}{v})$.

2. The p.g.f. of N given $\underset{\sim}{Z} = (Z_1,\ldots,Z_k)$, $h_{\underset{\sim}{z}}(u)$, say, is given by

$$h_{\underset{\sim}{z}}(u) = G_{\underset{\sim}{v}}^{(\underset{\sim}{z})}(u,\underset{\sim}{0})/G_{\underset{\sim}{v}}^{(\underset{\sim}{z})}(1,\underset{\sim}{0})$$

where

$$G_{\underset{\sim}{v}}^{(\underset{\sim}{z})}(\alpha,\underset{\sim}{\beta}) = \frac{\partial^{z_0} G(u,\underset{\sim}{v})}{\partial_{v_1}^{z_1}\ldots\partial_{v_k}^{z_k}}\Bigg]_{\substack{u=\alpha,\\ \underset{\sim}{v}=\underset{\sim}{\beta}}} \quad z_0 = z_1+\cdots+z_k \ .$$

Some general results, Cacoullos and Papageorgiou [9], concern the conditional distribution of N given $Z=z$, when $d=1$, i.e., when Z is a scalar r.v. It is shown that (for details we refer to [9]):

(i) $N|Z=z$ is a convolution of $N|Z=0$ and another nonnegative r.v. Y.

(ii) If N and X_1 have power series distributions (PSD), then the conditional distribution $N|Z=0$ is the same distribution of N with a new parameter $b_0\theta$ where $\theta = \theta_1/f_2(\theta_2); \theta_1$ is the parameter for N, and $f_2(\theta_2) = \sum_{k \geq 0} b_k\theta_2^k$ is the series function for X_1.

(iii) If N is Poisson and X_1 as in (ii), then the r.v. Y of has a PSD with series function the exponential polynomial $Y_z(b_1\theta,2!b_2\theta,\ldots,z!b_z\theta)$, that is, a general combinatorial type distribution, Harper [21].

Fitting some actual data concerning injury accidents (N) and fatal accidents or fatalities (Z) in eastern Virginia, Leiter and Hamdan [27] used a Poisson-Bernoulli and a Poisson-Poisson model. Negative binomial-Bernoulli or negative binomial-Poisson models gave more satisfactory results, as judged by the χ^2-criterion, Cacoullos and Papageorgiou [10]. A Poisson-binomial model [8], was also fitted to the same accident data.

In addition to the probabilistic aspects of these models, inferential problems are also examined. Thus, in view of the difficulty in obtaining explicit solutions of relevant maximum likelihood equations, special ad hoc

procedures are employed. Such are the methods of "even points", "zero frequencies" and "ratio of frequencies".

The method of "even points" was used, [10], to estimate the three parameters $(N, P$ and $\lambda)$ in the NB-P model; in addition to the estimators $\bar{X}$ and $\bar{Z}$ of $E(X)=NP$ and $E(Z)$, respectively, use is made of the equation

$$G(1,1)+G(-1,-1) = 2(P_{ee}+P_{00})$$

where $G(\cdot,\cdot)$ is the p.g.f. of (X,Z) and

$$P_{ee} = P[X=\text{even}, Z=\text{even}], \quad P_{00}=P[X=\text{odd}, Z=\text{odd}].$$

This yields a third estimating equation

$$1+\left[Q+Pe^{-2\lambda}\right]^{-N} = 2(S_{ee}+S_{00})/n \qquad (Q = 1+P)$$

where S_{ee} and S_{00} are the observed frequencies of $[X=\text{even}, Z=\text{even}]$ and $[X=\text{odd}, Z=\text{odd}]$ in a sample of size n, respectively.

The method of "zero frequencies" uses the relative frequency f_{00} in the $(0,0)$ cell and the proportions $f_{0.}$ and $f_{.0}$ of zero observations in the two marginals. The method of "ratios of frequencies" makes use of ratios such as $f_{1,0}/f_{00}, f_{1.}/f_{0.}$ etc. For further details, we refer to [9] and [10].

3. MULTIPARTITIONAL POLYNOMIALS

Several situations (cf. Section 2) call for the study of a compound (generalized) vector random variable (r.v.)

$$\underset{\sim}{Y} = \underset{\sim}{X_1}+\cdots+\underset{\sim}{X_N} \tag{1}$$

where $\underset{\sim}{X_1}, \underset{\sim}{X_2},\ldots$ are independent observations on the r.v. $\underset{\sim}{X}$ with probability

generating function (p.g.f.) $g(\underset{\sim}{v})$, say, and N is a non-negative integer-valued r.v. independent of the $\underset{\sim}{X}_i$, with p.g.f. $f(u)$. Then the compound (generalized) distribution of $\underset{\sim}{X}$ has p.g.f. $f(g(\underset{\sim}{v}))$. If the $\underset{\sim}{X}_i$ are continuous r.v.'s with common moment generating function (m.g.f.) (or characteristic function) $\varphi(\underset{\sim}{v})$, then the m.g.f. (or c.f.) of $\underset{\sim}{X}$ is again a composite function, $f(\varphi(\underset{\sim}{v}))$. Thus we could consider the usual properties of $\underset{\sim}{X}$ (probability function, moments, etc.) in terms of the functions f and g.

The case of scalar r.v.'s X_i led very naturally to the use of Bell (partition) polynomials Y_n, Charalambides [13], since Y_n may be regarded as the n-th derivative of the composite function $A(v)=f(g(v))$ in terms of the derivatives f_k of $f(.)$ and g_k of $f(.)$, $k=1,2,\ldots$. Similar considerations motivated the introduction of bipartitional polynomials A_{mn} for the study of $\underset{\sim}{Y}$ in the bivariate discrete case, Charalambides [14]. Essentially, the analogous multipartitional polynomials can be used for the treatment of multivariate distributions of $(N, \underset{\sim}{Y})$ (cf. Cacoullos and Papageorgiou [9]) since the p.g.f. of $(N, \underset{\sim}{Y})$ is $f(u\, g\,(\underset{\sim}{v}))$ as stated in Section 2.

For distributional purposes, it is convenient to define a bipartitional polynomial Y_{mn} in terms of derivatives as follows .

Let $A(u,v)=f(g(u,v))$ and set

$$f_k= \frac{d^k}{dt^k} f(t)\Big|_{t=g(u,v)}, \quad g_{ij}= \frac{\partial^i}{\partial u^i} \frac{\partial^i}{\partial v^i} g(u,v), \quad A_{mn}= \frac{\partial^m}{\partial u^m} \frac{\partial^n}{\partial v^n}A(u,v)=D_u^m D_v^n A(u,v)$$

Then

$$A_{mn}=Y_{mn}=Y_{mn}(f;g_{01},g_{10},\ldots,g_{mn})=\sum \frac{m!n!\, f_k}{k_{02}!k_{10}!\cdots k_{mn}!} \prod_{i=0}^{m} \prod_{j=0}^{n} \left(\frac{g_{ij}}{i!j!}\right)^{k_{ij}} \quad (2)$$

where the summation extends over all partitions of the bipartite indexes (mn), i.e., over all non-negative integers k_{ij} satisfying $\sum_{i=1}^{m} i \sum_{j=0}^{n} k_{ij}=m$, $\sum_{j=1}^{n} j \sum_{i=0}^{m} k_{ij}=n$; k is the number of parts in the partition. The expression on the R.H.S. of (2) is the analogue of di Bruno's formula for Bell (partition)

polynomials, Riordan [28], and may be used as an alternative definition of Y_{mn} (cf. (2) of Section 1).

An immediate consequence of the fact that $A_{mn} = Y_{mn}$ is that the probability function of $Y = (Y_1, Y_2)$, with p.g.f. $f(g(u,v))$, is

$$P_{m,n} = P[Y_1 = m, Y_2 = n] = \frac{1}{m!n!} Y_{mn}(f; P_{01}, P_{10}, \ldots, i!j! P_{ij}, \ldots, m!n! P_{mn}) \tag{3}$$

$$f_k = \frac{d^k}{dt^k} f(t)\Big|_{t=P_{00}}, \quad P_{ij} = P[X_1 = i, X_2 = j], \quad X = (X_1, X_2), \quad P_{00} = f(g(0,0)) = f(P_{00}).$$

Moreover, the factorial moments $\mu_{(m,n)} = E\left[(Y_1)_m (Y_2)_n\right]$ are

$$\mu_{(m,n)} = Y_{mn}(\alpha; \beta_{(0,1)}, \beta_{(1,0)}, \ldots, \beta_{(m,n)}) \quad \alpha_k \equiv \alpha_{(k)} \tag{4}$$

with $\alpha_{(k)}$ and $\beta_{(i,j)}$ denoting the factorial moments of N and (X_1, X_2), respectively.

As in the case of simple Bell (unipartitional) polynomials, using the umbral (Blissard) calculus, we may define the polynomials Y_{mn} of (2) in terms of their exponential generating function :

$$Y(u,v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Y_{mn} \frac{u^m}{m!} \frac{v^n}{n!} = \exp\left[f\{G(u,v) - G(0,0)\}\right] \tag{5}$$

with $f^k \equiv f_k$, $G(u,v) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g_{ij} \frac{u^i}{i!} \frac{v^j}{j!}$.

This can be used to derive recurrence relations for Y_{mn}. Moreover (5) (see also (4) and (5) of Section 1) implies the following important.

Remark. Bell polynomials can be used equally well both for truncated and non-truncated versions of discrete compound distributions. Truncation amounts to the cancellation of certain g_{ij}, corresponding to truncated values of the r.v.'s.

Bipartitional polynomials can arise in another situation of bivariate compounding. Let

$$Y_i = X_{i1} + X_{i2} + \cdots + X_{iN_i} \qquad i=1,2, \tag{6}$$

where N_1, N_2 are non-negative integer-valued r.v.'s with joint p.g.f. $F(\cdot, \cdot)$, the X_{ij} are independent with p.g.f.'s $g_i(\cdot)$ $(i=1,2)$ and the $\{X_{ij}\}$ are independent of (N_1, N_2). Then the p.g.f. G of (Y_1, Y_2) is easily seen to be the composite function

$$G(u,v) = F(g_1(u), g_2(v)).$$

The special case $f_k=1$ $(k=0,1,..)$ in (5) gives the bivariate (bipartitional) analogues $E_{mn}(g_{01}, g_{10}, g_{11}, \ldots, g_{mn})$ of Bell exponential polynomials (cf. (2) and (3) of Section 1). Thus in the important case of <u>compound bivariate Poisson</u> distributions, when either N or (N_1, N_2) is Poisson, the p.g.f. of (Y_1, Y_2) takes the form

$$G(u,v) = \exp\left[\lambda\{g(u,v)-1\}\right] \tag{7}$$

in the former case (1), whereas in the latter case (6), it is of the form

$$G(u,v) = \exp\left[\lambda_1\{g_1(u)-1\} + \lambda_2\{g_2(v)-1\} + \lambda_{12}\{g_1(u)g_2(v)-1\}\right] \tag{8}$$

Two examples of (7) and (8) have been studied by Charalambides and Papageorgiou [17]: (a) X_i in (1) is a bivariate binomial i.e. $g(u,v) = (p_{00} + p_{10}u + p_{01}v + p_{11}uv)^n$ and (b) the X_{ij} in (6) are independent binomials, i.e., $g_i(u) = (p_i u + q_i)^{n_i}$. They provide alternatives to Neyman type A models used by Holgate [22] to fit certain ecological data.

It should be observed that the exponential polynomials E_{mn} are associated with compound bivariate distributions in which (Y_1, Y_2) has a p.g.f. of the form

$$G(u,v) = e^{h(u,v)}. \tag{9}$$

In this situation certain useful relations may be stated.

The bipartitional exponential polynomials $E_{mn}(g_{01}, g_{10}, \ldots)$ satisfy

the recurrence

$$E_{m,n+1} = \sum_{s=0}^{n} \sum_{r=0}^{m} \binom{m}{r}\binom{n}{s} g_{r,s+1} Y_{n-r+1,n-s} \; , \quad E_{00}=1 \; . \tag{10}$$

The p.f. $P(m,n;h)=P\left[Y_1=m,Y_2=n\right]$ associated with (9) is given by (cf (3))

$$P(m,n;h)=e^{h(0,0)} E_{m,n}(h_{01},h_{10},h_{11},\ldots,h_{mn})/m!n!$$

where (see (1) of Section 1)

$$h_{rs} = D_v^s D_u^r h(u,v)\big|_{u=0,\ v=0} \; .$$

As regards the factorial $\mu(m,n;h)$ of (Y_1,Y_2) we have (cf (4))

$$\mu(m,n;h) = E_{m,n}(c_{01},c_{10},c_{11},\ldots,c_{mn})$$

where

$$c_{rs} = D_v^s D_u^r h(u,v)\big|_{u=1,v=1} \; .$$

Using (10) we get the recurrences

$$P(m,n+1;h) = \frac{1}{n+1} \sum_{s=0}^{n} \sum_{r=0}^{m} \frac{h_{r,s+1}}{r!s!} P(m-r,n-s;h) \; ,$$

$$\mu(m,n+1;h) = \sum_{s=0}^{n} \sum_{r=0}^{m} \binom{m}{r}\binom{n}{s} c_{r,s+1} \mu(m-r,n-s;h).$$

Marginal and conditional p.f.'s can also be given in terms of E_{mn}. Thus

$$P\left[Y_1=m\right] = e^{h(0,1)} E_m(h_1,\ldots,h_m)/m!$$

where E_m is the corresponding unipartitional polynomial and

$$h_r = D_u^r h(u,1)\big|_{u=0} \qquad r=1,\ldots,m$$

and the conditional p.f. of Y_2 given $Y_1=m$ is given by

$$P\left[Y_2=n\,\middle|\,Y_1=m\right] = \exp\left[h(0,0)-h(0,1)\right] \frac{E_{m,n}(h_{01},h_{10},\ldots,h_{mn})}{n!E_m(h_1,\ldots,h_m)}$$

For more details we refer to $[14]$.

4. ASYMPTOTICS OF COMBINATORIAL DISTRIBUTIONS

Relations between Stirling and C-numbers and Bell polynomials, as well as their role in discrete-distribution theory and occupancy-type problems have already been made clear in the preceding sections.

Another interesting probabilistic aspect of Stirling and C-numbers is tied up with the asymptotic normality of the so called combinatorial distributions. A first result in this direction is that of Harper [21] showing the asymptotic normality of the combinatorial distribution defined in terms of Stirling numbers of the second kind, $S(m,n)$, namely, the distribution of

$$P[X_m=n] = \frac{S(m,n)}{B_m} \qquad n=0,\ldots,m, \quad m=0,1,\ldots \qquad (1)$$

where $B_m = \sum_{n=0}^{m} S(m,n)$ is the Bell number. Another case considered by Charalambides, [12], is when $S(m,n)$ is replaced by the C-number $C(m,n,s)$, Cacoullos & Charalambides [6]. In a general combinatorial distribution, the $S(m,n)$ are replaced by $A(m,n)$ which are assumed to satisfy the "generalized Pascal triangle":

$$A(m+1,n)=g(m,n)A(m,n)+h(m,n)A(m,n-1) \qquad (2)$$

where g and h are positive; also the r.h.s of (1) is multiplied by λ^n, i.e. the general combinatorial distribution is defined by

$$P[X_m=n] = \frac{A(m,n)}{A_m(\lambda)} \lambda^n \quad , \quad A_m(\lambda) = \sum_{n=0}^{m} A(m,n)\lambda^n . \qquad (3)$$

The question raised here, Kyriakousis [26], is under what conditions on $g(m,n)$ and $h(m,n)$ the asymptotic normality (as $m \to \infty$) of the general combinatorial distribution obtains.

In particular, the problem of asymptotic normality is studied through the asymptotic behavior of the polynomials

$$A_m(\lambda) = \sum_{n=0}^{m} A(m,n)\lambda^n .$$

It is assumed that the $A_m(\lambda)$ are Bell exponential polynomials, defined by their exponential generating function :

$$\sum_{m=0}^{\infty} A_m(\lambda) \frac{z^m}{m!} = \exp\{\lambda[f(z)-f(0)]\}, \quad f(z) = \sum_{n \geqslant 0} c_n z^n.$$

For example, if $f(z)=e^z$, then $A_m(1)=B_m$, the so-called Bell number and $A(m,n)=S(m,n)$; if $f(z)=(1+z)^s$, $s>0$, then $A(m,n)=C(m,n,s)$, the C-numbers. Furthermore, asymptotic expressions for ratios $A_{m+k}(\lambda)/A_m(\lambda)$ are obtained under certain conditions on g. These ratios determine the asymptotic behavior of the variance $(k=2)$ and the mean $(k=1)$ of the X_{mn}. Then applying the normal convergence criterion to the sequence $\{X_{mn}\}$ shows the asymptotic normality of the corresponding combinatorial distribution. Also, a useful result in this direction is that of Haigh [20], which requires to show that the polynomial p.g.f. of X_m has real roots.

Some general conditions on $g(m,n)$ and $h(m,n)$ under which the corresponding combinatorial distributions converge to the normal (as $m \rightarrow \infty$) are given in Kyriakousis [26]. They imply, as special cases, the asymptotic normality of the combinatorial distributions defined by :

(a) the signless (absolute) Stirling numbers of the first kind $|s(m,n)|$, which satisfy the recurrence (cf. Appendix A)

$$|s(m+1,n)|=m|s(m,n)|+|s(m,n-1)|,$$

(b) the binomial coefficients $\binom{m}{n}$ so that by taking $\lambda=p/q$ in (3) one obtains the well-known asymptotic normality of the binomial distribution

(c) the Eulerian numbers

$$E(m+1,n)=n\ E(m,n)+(m-n+2)E(m,n-1)$$

and the generalized Eulerian numbers, Dwyer [19], $E_a(m,n)$, a>0 integer, with

recurrence

$$E_a(m+1,n)=(n+a)E_a(m,n)+(m-n+2-a)E_a(m,n),$$

(d) the Stirling numbers $S(m,n)$ of the second kind $(a_1(m)=b_0(m)=1,$ $a_0(m)=0)$, first shown by Harper $|21|$ for $\lambda=1$; also the C-numbers, $C(m,n,s)$, and the signless numbers $|C(m,n,-s)|$, [12] (see also Appendix A). Finally, the non-central Stirling numbers of the second kind $S_a(m,n)$, studied by Koutras [25], with the recurrence (see Section 5)

$$S_a(m+1,n) = (n-a)S(m,n)+S_a(m,n-1),$$

for $a<0$ also converge to normality

5. NON-CENTRAL STIRLING NUMBERS - MULTIPARAMETER STIRLING AND C-NUMBERS.

The Stirling numbers of the first kind $s(n,k)$ and the second kind $S(n,k)$ are usually defined ϵ the coefficients in the expansion of the factorial $(x)_n$ in powers of x and vice versa (see also Appendix A). The non-central ones are their analogues when x^k is replaced by $(x-c)^k$ for some real c. This leads to an equivalent definition in terms of exponential generating functions (egf). Thus the egf for the non-central (around c) numbers of the first kind $s_c(n,k)$ is found to be

$$f_k(t) = \sum_{n=k}^{\infty} s_c(n,k) \frac{t^n}{n!} = (1+t)^c \left|\frac{1}{k!} \log(1+t)\right|^k; \qquad (1)$$

similarly the egf $h(\cdot)$ of $S_c(n,k)$ is given by

$$h_k(t) = e^{-ct} \frac{1}{k!} (e^t-1)^k. \qquad (2)$$

These expressions can be used to obtain relations between non-central and central (c=0) Stirling numbers. Moreover, they are con-

venient for the definition of generalized non-central Stirling numbers $s_c(n,k,r)$ and $S_c(n,k,r)$, by subtracting the first r terms in the expansions of $\log(1+t)$ and e^t, respectively.

Applications in probability and occupancy problems.

Considering power series distributions, $(1-t)^c$, $c<0$, is the series function of a negative binomial and $-\log(1-t)$ the series function of a logarithmic series distribution; it can be concluded from (1) that the "signiless" non-central numbers of the first kind, $|s_c(n,k)| = (-1)^{n-k}s_c(n,k)$ are associated with the convolution of a negative binomial and a k-fold convolution of a logarithmic series distribution; similarly the generalized $s_c(n,k,r)$ correspond to left-truncated logarithmic distributions.

Another use in probability theory of $s_c(n,c)$ is in expressing the factorial moments π_n of a random variable X in terms of its moments $\mu_{k,c}$ about the point c, that is,

$$\pi_n = \sum_{k=0}^{n} s_c(n,k)\mu_{k,c} .$$

Thus taking $c=\mu=E(X)$, π_n is expressed in terms of the central moments μ_k. Conversely, the $S_c(n,k)$ can be used to express $\mu_{n,c}$ in terms of π_n. From the point of view of distribution theory, the $S_c(n,k)$ are associated with convolutions of usual and zero-truncated Poisson distributions.

An occupancy-type interpretation of $S_c(n,k)$ is the following : n distringuishable balls can be distributed into a set of k identical cells and j distinguishable boxes so that every cell is occupied in $S_{-j}(n,k)$ ways. Similarly for $S_c(n,k,r)$.

For additional results and details we refer to Koutras [25].

Multiparameter Stirling and C-numbers.

Another extension of the usual Stirling and C-numbers in a different direction is motivated by the estimation problems when several independent

samples are available from the same parent distribution but the truncation points differ from sample to sample. The relevant details of applications and occupancy-type interpretations are given in Appendix A.

6. BELL POLYNOMIALS IN FLUCTUATION THEORY

Bell (exponential) polynomials have been used in the study of generalized (compound) discrete distributions, due to their interpretation as derivatives of composite functions (Sections 1 through 4). Another important area of probability theory where such polynomials provide a powerful tool is the so called fluctuation theory as developed mainly by E.S. Andersen and W. Feller.

Consider a generalized random walk $\{S_k\}$, where $S_k = X_1 + \cdots + X_k$ $k=1,2,\ldots$ $(S_0 \equiv 0)$ and the X_i are i.i.d. r.v.'s. Using a classical result of Touchard $|32|$ on the number of permutations of n elements with specified numbers of cycles possessing certain properties, in conjunction with Spitzer's combinatorial lemma, yields a simple combinatorial proof of the basic result:

$$p_n \equiv P[S_1 \; 0,\ldots,S_n > 0] = \frac{1}{n!} C_n(a_1,\ldots,a_n)$$

$$q_n \equiv P[S_1 < 0,\ldots,S_n < 0] = \frac{1}{n!} C_n(1-a_1,\ldots,1-a_n)$$

with $a_k = P[S_k > 0]$, $k=1,2,\ldots,n$ $(n=1,2,\ldots)$ and C_n denoting the cycle indicator function , Riordan [28], related to Bell polynomials $Y_n(t_1,\ldots,t_n)$ by :

$$C_n(t_1,\ldots,t_n) = Y_n(t_1,t_2,2!t_3,\ldots, (n-1)!t_n).$$

Further simplifications are obtained by introducing the polynomials, [16],

$$C_{k,n}(x,y) \equiv C_{k,n}(x_1,\ldots,x_n,y_1,\ldots,y_n) = \binom{n}{k} C_k(x) C_{n-k}(y).$$

Exploiting certain properties of $C_{k,n}(x,y)$, one can show the basic result

$$P[N_n = k] = p_k q_{n-k} = \frac{1}{n!} C_{k,n}(a_1,\ldots,a_n,1-a_1,\ldots,1-a_n)$$

and hence, e.g., the recurrence

$$P[N_{n+1}=k] = \frac{1}{n-1+k} \sum_{r=0}^{n-k} (1-a_{r+1}) P[N_{n-r}=k] ,$$

N_n denoting the number of positive partial sums S_k, $k=1,\ldots,n$.

Also simplified proofs can be obtained of some results concerning symmetrically dependent (exhangeable) r.v.'s after proving a result along the lines of Spitzer's combinatorial lemma. For details we refer to [16].

7. CHARACTERIZATIONS OF COMPOUND DISTRIBUTIONS BY REGRESSION AND BELL-TYPE POLYNOMIALS AND NUMBERS.

So far Bell-type polynomials were discussed in relation to the distribution of a generalized (compound) discrete random variable (rv); also in generalized random walks (fluctuation theory). In the latter case, it was indicated how Bell polynomials can be extended to provide simple proofs for some known basic results. In the former case, the probability generating function of the generalized rv $Y=XvZ$,

$$Y = Z_1+Z_2+\cdots+Z_X, \tag{1}$$

can be expressed in terms of Bell-type polynomials. For example, Stirling type polynomials (with coefficients Stirling numbers of the second kind) appear whenever the generalizing variable Z is a Poisson and C-type polynomials (with coefficients C-numbers) whenever the Z-variable is either a binomial or a negative binomial, [9].

Bell-type polynomials can also be used to express the regression (posterior mean) $m(y) = E[X|Y=y]$ of a compound mixing r.v. X on the mixture variable Y, with (absolute) probability function (p.f.)

$$p(y) = \sum_x p(y|x) f(x) \tag{2}$$

(p(·|x) denoting the conditional p.f. of Y(X) given X=x and f(·) the prior p.f. of X). This came up in specific applications of the following general characterization result.

Let p(y|x) in (2) be a binomial, negative binomial or Poisson for every x=1,2,... . Then m(y) characterizes both p(·) and f(·). (The details are given in Appendix B, exhibiting also several examples of Bell-type polynomials and numbers).

In fact, m(0) alone is sufficient to characterize p(·) and f(·) provided p(·|x) is an x-fold convolution of the same r.v. Z for every s = P[Z=0]>0, Cacoullos [5]. Clearly, by varying arbitrarily the X-distribution, an infinite variety of bivariate discrete distributions can be characterized in this fashion. The result is based on the fact that the p.g.f. h of X satisfies the differential equation

$$m(0) = m(0;s) = sh'(s)/h(s) \quad 0<s<1.$$

Hence f(·) is determined and by (2) also p(·).

Finally, characterizations in terms of m(y) are obtained for continuous analogues of (2), when X and Y are continuous. Here the role of X is played by a continuous parameter θ, which is itself a r.v. with some prior density f(·). However, since these results fall rather outside the main scope of this research, no further details are given here.

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APPENDIX A

MULTIPARAMETER STIRLING AND C-NUMBERS: RECURRENCES AND APPLICATIONS*

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ABSTRACT

Multiparameter Stirling and C-numbers are defined via exponential generating functions and basic recurrence relations are given; also, some combinatorial and occupancy type interpretations are provided.

Recurrence relations are derived for certain ratios of simple, generalized and multiparameter Stirling and C-numbers. These recurrences are useful in the computation of minimum variance unbiased estimates (mvue) for classical discrete distributions truncated on the left. Asymptotic relations between these numbers are also included.

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1. INTRODUCTION

The Stirling numbers of the first and second kind are less known among statisticians than among people dealing with combinatorics or finite differences. Only recently have they made their appearance in distribution theory and statistics. They emerge in the distribution of a sum of zero-truncated classical discrete distributions: those of the second kind, $S(m,n)$, in the case of a Poisson distribution truncated away from zero, Tate and Goen (1958), Cacoullos (1961); the signless (absolute-value) Stirling numbers of the first kind, $|s(m,n)|$, in the logarithmic series distribution, Patil (1963). In general, such distributional problems are essential in the construction of minimum variance unbiased estimators (mvue) for parametric functions of a left-truncated power series distribution (PSD).

Analogous considerations for binomial and negative binomial distributions truncated away from zero motivated the introduction of a new kind of numbers, called C-numbers by Cacoullos and Charalambides (1975). These three-parameter C-numbers, $C(m,n,k)$, were further studied by Charalambides (1977), who gave the representation $C(m,n,k) = \sum_{r=n}^{m} k^r s(m,r) S(r,n)$ in terms of Stirling numbers of the first kind, $s(m,r)$, and the second kind $S(r,n)$. Interestingly enough, this representation in a disguised form was, in effect, used by Shumway and Gurland (1960) to tabulate C-numbers, involved in the calculation of Poisson-binomial probabilities.

The so-called generalized Stirling and C-numbers emerged as a natural extension of the corresponding simple ones in the study of the mvue problem for a PSD truncated on the left at an arbitrary (known or unknown) point Charalabmides (1974b). It should be mentioned that, in particular, the generalized Stirling numbers of the second kind were independently rediscovered and tabulated by Sobel et. al. (1977), in connection with the Incomplete Type I - Dirichlet integral.

The multiparameter Stirling and C-numbers are the analogues of generalised Stirling and C-numbers in a multi-sample situation where the underlying PSD is multiply truncated on the left, Cacoullos (1975), (1977).

Recurrence relations for ratios of Stirling and C-numbers are necessary, because the mvue of certain parametric functions of left-truncated logarithmic series, Poisson, binomial and negative binomial distributions are expressed in terms of such ratios. These recurrences bypass the computational difficulties which come from the fact that the numbers themselves (but not the ratios of interest) grow very fast with increasing arguments. Recurrences for ratios of simple Stirling numbers of the second kind were developed by Berg (1975).

The main purpose of this paper is to provide recurrences for certain ratios of multiparameter Stirling and C-numbers, thus unifying several special results, including those of Berg (1975). For the development of the topic, we found the use of exponential generating functions (egf) most appropriate, both for introducing the numbers themselves and deriving recurrences as well. Without claiming completeness, we included certain basic recurrences. As observed elsewhere, Cacoullos (1975) (1977), it is emphasised here, once more, that in the study of PSD's the egf approach is the one suggested by the probability function itself in its truncated form. Also, we found it appropriate to include certain asymptotic relations between Stirling and C-numbers, which reflect corresponding relations between binomial and Poisson distributions or logarithmic series and negative binomial distributions.

A typical result, which involves ratios considered here, is the following. Let x_{ij}, $j=1,\ldots,n_i$ be a random sample from a left-truncated one-parameter PSD distribution with p.f.

$$p(x;\theta) = \frac{a_i(x)\theta^x}{f_i(\theta,r_i)} , \quad x=r_i, r_i+1,\ldots \tag{1.1}$$

where $f_i(\theta,r_i) = \sum\limits_{x=r_i}^{\infty} a_i(x)\theta^x$, $i=1,\ldots,k$. If the truncation point $\underline{r}=(r_1,\ldots,r_k)$ is known and $a_i(x)>0$ for every $x>r_i$, $i=1,\ldots,k$, then, Cacoullos (1977), for every $j=1,2,\ldots$, θ_j is estimable and its (unique) mvue, based on all k independent samples $\{x_{ij}\}$, is given by

$$\hat{\theta}_j(m) = (m)_j \frac{a(m-j;\underset{\sim}{n},\underset{\sim}{r})}{a(m;\underset{\sim}{n},\underset{\sim}{r})} \tag{1.2}$$

where $\underset{\sim}{n}=(n_1,\ldots,n_k)$, $\underset{\sim}{r}=(r_1,\ldots,r_k)$, $(m)_j=m(m-1)\cdots(m-j+1)$ and

$$a(m;\underset{\sim}{n},\underset{\sim}{r}) = \frac{m!}{n_1!\ldots n_k!} \sum \prod_{i=1}^{k} \prod_{j=1}^{n_i} a_i(x_{ij}), \tag{1.3}$$

where the summation extends over all ordered N-tuples $(N=n_1+\cdots+n_k)$ of integers x_{ij} satisfying $x_{ij} \geq r_i$, $\sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}=m$. In the cases of interest (Poisson, binomial, etc), the numbers (integers) $a(m;\underset{\sim}{n},\underset{\sim}{r})$ turn out to be Stirling or C-numbers, depending on the series function f_j in (1.1), which at the same time, suggests the corresponding egf of these numbers.

2. MULTIPARAMETER STIRLING NUMBERS OF THE FIRST KIND: DEFINITION-GENERAL PROPERTIES.

Let $r_1,\ldots,r_k$ and $n_1,\ldots,n_k$ be non-negative integers $(k \geq 1)$. The multiparameter Stirling numbers of the first kind with parameters $\underset{\sim}{r}=(r_1,r_2,\ldots,r_k)$ and $\underset{\sim}{n}=(n_1,n_2,\ldots,n_k)$, to be denoted by $s(m;\underset{\sim}{n},\underset{\sim}{r})$, can be defined (cf. Cacoullos, 1975) by the egf

$$\mathcal{E}_{\underset{\sim}{n}}(t;\underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} s(m;\underset{\sim}{n},\underset{\sim}{r}) \, t^m/m! = \prod_{i=1}^{k} \frac{1}{n_i!} \left[\log(1+t) - \sum_{j=1}^{r_i-1} (-1)^{j-1} \frac{t^j}{j} \right]^{n_i} \tag{2.1}$$

where we set $m=\underset{\sim}{r}'\underset{\sim}{n} = r_1 n_1+\cdots+r_k n_k$.

The special case $k=1$, $r_1=r$, $n_1=n$ yields the generalized Stirling numbers of the first kind, $s(m;n,r)$, defined by Charalambides (1974a), while $k=1$, $r=1$ gives the simple Stirling numbers of the first kind, $s(m,n)$. Propositions 2.1-2.3 summarize basic properties and recurrences for $s(m;\underset{\sim}{n},\underset{\sim}{r})$ and facilitate their computation.

Remark 2.1. In the sequel, in order to avoid unnecessary complications in the recurrences, we assume that all $n_j>0$; some n_j, say ν, are zero, then the parameter k becomes $k'=k-\nu$ and the necessary modifications are obvious.

<u>Proposition 2.1</u>. The multiparameter Stirling numbers of

the first kind $s(m;\underline{n},\underline{r})$ have the following representation

$$s(m;\underline{n},\underline{r}) = (-1)^{m-N} \frac{m!}{n_1! \dots n_k!} \sum_m \prod_{i=1}^{k} \prod_{j=1}^{n_i} \frac{1}{x_{ij}} \qquad (2.2)$$

where $N = n_1 + \dots + n_k$ and the summation extends over all ordered N-tuples

of integers x_{ij} satisfying the relations

$$x_{ij} \geq r_i , \; i = 1,\dots,k \quad \text{and} \quad \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} = m.$$

<u>Proof</u>. We have

$$\gamma(t,r_i) = \log(1+t) - \sum_{k=1}^{r_i-1} (-1)^{k-1} \frac{t^k}{k} = \sum_{k=r_i}^{\infty} (-1)^{k-1} \frac{t^k}{k} , \; i=1,\dots,k. \quad (2.3)$$

Forming the Cauchy product of series, we find, in virtue of (2.1),

$$g_{\underline{n}}(t;\underline{r}) \prod_{i=1}^{k} n_i! = \prod_{i=1}^{k} \left[\gamma(t,r_i)\right]^{n_i} = \sum_{m=\underline{r}'\underline{n}}^{\infty} (-1)^{m-N} t^m \sum_m \prod_{i=1}^{k} \prod_{j=1}^{n_i} \frac{1}{x_{ij}} \quad (2.4)$$

where $\sum_m$ has the same meaning as above.

Comparing (2.4) with (2.1) we get (2.2).

To obtain recurrence relations, we make use of the easily verified

difference-differential equation, satisfied by the egf $g_{\underline{n}}(t,\underline{r})$ in (2.1),

namely,

$$(1+t) \frac{d}{dt} g_{\underline{n}}(t;\underline{r}) = \sum_{i=1}^{k} (-1)^{r_i-1} t^{r_i-1} g_{\underline{n}-\underline{e}_i}(t;\underline{r}) \qquad (2.5)$$

where $\underline{e}_i = (0,\dots,0,1,0,\dots,0)$, i.e., a k-component vector with zero

components except the i-th component which is equal to one.

Proposition 2.2.: $(m,\underline{n})$ - wise relations: The numbers $s(m,\underline{n},\underline{r})$ satisfy the

recurrence relation

$$s(m+1;\underline{n},\underline{r}) + m \, s(m;\underline{n},\underline{r}) = \sum_{i=1}^{k} (-1)^{r_i-1} (m)_{r_i-1} s(m-r_i+1;\underline{n}-\underline{e}_i,\underline{r}) \qquad (2.6)$$

with initial conditions

$s(0;0,\underline{r}) = 1$, $s(0;\underline{n},\underline{r}) = 0$ whenever $\sum_{i=1}^{k} r_i n_i > 0$, $s(m;\underline{n},\underline{r}) = 0$ if $m < \underline{r}'\underline{n}$.

<u>Proof</u>. Equation (2.5) in virtue of (2.1) can be written as

$$(1+t) \sum_{n=\underline{r}'\underline{n}}^{\infty} s(m;\underline{n},\underline{r}) \frac{t^{m-1}}{(m-1)!} = \sum_{i=1}^{k} \sum_{m=\underline{r}'\underline{n}-r_i}^{\infty} (-1)^{r_i-1} s(m;\underline{n}-\underline{e}_i,\underline{r}) \frac{t^{m+r_i-1}}{m!} \quad (2.7)$$

Equating the coefficients of $t^n/m!$ in (2.7) yields (2.6). Note that equation (2.6) for $k = 1$, $r_1 = 1$ gives the well-known recurrence for the simple Stirling numbers of the first kind

$$s(m+1,n) = s(m,n-1) - m\, s(m,n). \qquad (2.8)$$

Proposition 2.3. $(m;\underline{n},\underline{r})$ - wise relations: The numbers $s(m;\underline{n},\underline{r})$ satisfy

$$s(m;\underline{n},\underline{r}+\underline{e}_i) = \sum_{j=0}^{n_i} (-1)^{jr_i} \frac{(m)_{jr_i}}{j!\,(r_i)^j} s(m-jr_i;\underline{n}-j\underline{e}_i,\underline{r}), \quad i=1,\ldots k \quad (2.9)$$

Proof. We have, using also (2.3)

$$g_{\underline{n}}(t;\underline{r}+\underline{e}_i) = \frac{1}{n_i!}\left[\gamma(t;\underline{r})+(-1)^{r_i}\frac{t^{r_i}}{r_i}\right]^{n_i} \prod_{\substack{j=1\\j\neq i}}^{k} \frac{1}{n_j!}\left[\gamma(t,r_j)\right]^{n_j} \qquad (2.10)$$

and using the binomial expansion

$$\left[\gamma(t,r_i)+(-1)^{r_i}\frac{t^{r_i}}{r_i}\right]^{n_i} = \sum_{j=0}^{n_i}\binom{n_i}{j}(n_i-j)!\, g_{n_j-j}(t,r_i)(-1)^{jr_i}\frac{t^{jr_i}}{r_i^j} \qquad (2.11)$$

we can write (2.10) as

$$\sum_{m=r'\underline{n}+n_j}^{\infty} s(m;\underline{n},\underline{r}+\underline{e}_i)\frac{t^m}{m!} = \sum_{j=0}^{n_i}\frac{(-1)^{jr_i}}{j!\,r_i^j}\sum_{m=r'\underline{n}-jr_i} s(m;\underline{n}-j\underline{e}_i,\underline{r})\frac{t^{m+jr_i}}{m!} \qquad (2.12)$$

Hence, equating the coefficients of $t^m/m!$ we obtain (2.9).

Signless multiparameter Stirling numbers. From the recurrence relation (2.6), it follows that the numbers $s(m;\underline{n},\underline{r})$ are integers. Moreover from the representation in (2.2), we conclude that $s(m;\underline{n},\underline{r})$ is an integer with sign $(-1)^{K-m}$, where $N = n_1+\ldots+n_k$. Therefore if we multiply (2.6) by $(-1)^{m-N+1}$, we obtain

$$|s(m+1;\underline{n},\underline{r})| = m|s(m;\underline{n},\underline{r})| + \sum_{i=1}^{k}(m)_{r_i-1}|s(m-r_i+1;\underline{n}-\underline{e}_i,\underline{r})| \qquad (2.13)$$

we call $|s(m;\underline{n},\underline{r})|$ the signless (positive) multiparameter (k-parameter) Stirling number of the first kind. We will show

Proposition 2.4. The egf of $|s(m,\underline{n},\underline{r})|$ is given by

$$g_{\underline{n}}^*(t;\underline{r}) = \sum_{m=r'\underline{n}}^{\infty} |s(m;\underline{n},\underline{r})|\frac{t^m}{m!} = \prod_{i=1}^{k}\frac{1}{n_i!}\left[-\log(1-t)-\sum_{j=1}^{r_i-1}\frac{t^j}{j}\right]^{n_i} \qquad (2.14)$$

Proof. From the difference equation (2.13) it is easily verified that the

egf $\varepsilon_n^*(t;\underline{r})$ satisfies the difference-differential equation

$$(1-t)\frac{d}{dt} h_n(t;\underline{r}) = \sum_{i=1}^{k} t^{r_i-1} \varepsilon_{n-e_i}^*(t;\underline{r}) \tag{2.15}$$

which in turn yields (2.14).

Alternatively, (2.14) leads to the representation of $|s(m;\underline{n},\underline{r})|$ as

obtained from (2.2).

3. RATIOS OF MULTIPARAMETER STIRLING NUMBERS OF THE FIRST KIND.

We define, as ratio of multiparameter Stirling numbers of the first

kind with respect to argument m, the function

$$R_1(m;\underline{n},\underline{r}) = \frac{s(m+1;\underline{n},\underline{r})}{s(m;\underline{n},\underline{r})} \tag{3.1}$$

Ratios with respect to the arguments n_j, r_j, $i = 1,\ldots,k$ can also

be defined. The main reason for considering ratios with respect to m

is seen from (1.1), which, actually, involves reciprocals of R_1, when

we are concerned with the parameter of a logarithmic series distribution.

Proposition 3.1. A recurrence relation for the ratio $R_1(m;\underline{n},\underline{r})$,

independent of the multiparameter Stirling numbers of the first kind,

is given by

$$R_1(m;\underline{n},\underline{r})+m = \frac{\displaystyle\sum_{j=1}^{k} \frac{(m)_{r_j-1} \, r_j n_j}{(\underline{r'n})_{r_j}} \prod_{i=1}^{r_j-r'n} R_1(m-r_j+1-i;\underline{n}-e_j,\underline{r})}{\displaystyle\prod_{i=1}^{m-r'n} R_1(m-i;\underline{n},\underline{r})} \tag{3.2}$$

for $n \geq 1$ and $m \geq r'n$, with the boundary conditions

$$R_1(m,1,r) = -m \tag{3.3}$$

and

$$R_1(r'n,n,r) = (-1)(r'n+1) \prod_{i=1}^{k} r_i^{n_i} \sum_{j=1}^{k} \frac{n_j}{r_j^{n_j-1}(r_j+1) \prod_{\substack{i=1 \\ i \neq j}}^{k} r_i^{n_i}} \tag{3.4}$$

<u>Proof.</u> Using equation (3.1), it can be easily seen that

$$\prod_{i=1}^{m-r'n} R_1(m-i;n,r) = \frac{s(m;n,r)}{s(r'n,n,r)} . \tag{3.5}$$

But equation (2.2), for $m = r'n$, $m_{i1} = m_{i2} = \ldots = m_{in_i} = r_i$, becomes

$$s(r'n,n,r) = (-1)^{r'n-N} \frac{(r'n)!}{\prod_{i=1}^{k} n_i! \prod_{i=1}^{k} r_i^{n_i}} . \tag{3.6}$$

Consequently, equation (3.5) becomes

$$s(m;n,r) = \frac{(-1)^{r'n-N}(r'n)! \prod_{i=1}^{m-r'n} R_1(m-i;n,r)}{\prod_{i=1}^{k} n_i! \prod_{i=1}^{k} r_i^{n_i}} , \tag{3.7}$$

From equations (2.6) and (3.1) we have

$$R_1(m;n,r)+m = \frac{\sum_{j=1}^{k}(-1)^{r_j-1}(m)_{r_j-1} s(m-r_j+1;n-e_j,r)}{s(m;n,r)} \tag{3.8}$$

and substituting for $s(m-r_j+1,n-e_j,r)$ and $s(m,n,r)$ from (3.7) yields (3.2).

By definition

$$R_1(r'n,n,r) = \frac{s(r'n+1,n,r)}{s(r'n,n,r)} , \tag{3.9}$$

using equation (2.2) for $m = r'n+1$, $m_{i1} = m_{i2} = \ldots = m_{i,l-1} = m_{i,l+1} = \ldots = m_{i,n_i} = r_j$, $m_{i1} = r_{i+1}$ for $l = 1,\ldots,n_i$, and equation (3.6), the required formula (3.4) is easily obtained.

The special case $k = 1$ yields

Proposition 3.2. A recurrence relation for the ratio $R_1(m,n,r)$, independent of the generalized Stirling numbers of the first kind, is given by

$$R_1(m,n,r)+m = \frac{\dfrac{rn\,(m)_{r-1}}{(rn)_r}\ \displaystyle\prod_{i=1}^{m+1-rn} R_1(m-r+1-i,n-1,r)}{\displaystyle\prod_{i=1}^{m-rn} R_1(m-i,n,r)} \tag{3.10}$$

for $n \geqslant 1$ and $m > rn$, with

$$R_1(m,1,r) = -m \tag{3.11}$$

$$R_1(rn,n,r) = -\frac{rn(rn+1)}{r+1} \tag{3.12}$$

Also for $k = 1$, $r = 1$ we obtain,

Proposition 3.3. A recurrence relation for the ratio $R_1(m,n)$, independent of the simple Stirling numbers of the first kind, is given by

$$R_1(m,n)+m = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_1(m-i,n-1)}{\displaystyle\prod_{i=1}^{m-n} R_1(m-i,n)} \tag{3.13}$$

for $n > 1$ and $m > n$, with

$$R_1(m,1) = -m \tag{3.14}$$

$$R_1(n,n) = - n(n+1)/2 \tag{3.15}$$

Proposition 3.4. An alternative recurrence relation for the ratio $R_1(m,n,r)$ is given by

$$R_1(m,n,r)+m = \frac{m}{m-r+1}\ \frac{\left[R_1(m-1,n,r)+m-1\right] R_1(m-r,n-1,r)}{R_1(m-1,n,r)} \tag{3.16}$$

for $n > 1$ and $m > rn$. $R_1(m,1,r)$ and $R_1(rn,n,r)$ are given by (3.11) and (3.12), respectively.

Proof. Using equation (2.6) with $k = 1$, we have

$$R_1(m,n,r)+m = \frac{(-1)^{r-1}\,(m)_{r-1}\,s(m-r+1,n-1,r)}{s(m,n,r)} \tag{3.17}$$

from which equation (3.16) can be easily derived.

Applying Proposition 3.4 with r=1 gives

Proposition 3.5. An alternative recurrence relation for the ratio $R_1(m,n)$ is given by

$$R_1(m,n)+m = \frac{\left[R_1(m-1,n)+m-1\right] R_1(m-1,n-1)}{R_1(m-1,n)}\qquad n>1,\quad m>n. \tag{3.18}$$

4. MULTIPARAMETER STIRLING NUMBERS OF THE SECOND KIND

The multiparameter Stirling numbers of the second kind $S(m;\underline{n},\underline{r})$

defined by their egf

$$f_{\underline{n}}(t;\underline{r}) = \sum_{m=\underline{r}'\underline{n}}^{\infty} S(m;\underline{n},\underline{r}) \frac{t^m}{m!} = \prod_{i=1}^{k} \frac{1}{n_i!} \left[e^t - \sum_{j=0}^{r_i-1} \frac{t^j}{j!} \right]^{n_i} \qquad (4.1)$$

Taking $k=1, r_1=r$ gives the generalized Stirling numbers of the second kind, $S(m,n,r)$
Charalambides (1974a), taking $k\equiv1$, $r=1$ defines the simple Stirling numbers
$S(m,n)$. The following properties of $S(m,n,r)$ can easily be established (cf.
Section 2).

a) They have the representation

$$S(m;\underline{n},\underline{r}) = \frac{m!}{n_1!\ldots n_k!} \sum_{m} \prod_{i=1}^{k} \prod_{j=1}^{n_i} \frac{1}{x_{ij}!} \qquad (4.2)$$

where the summation extends over all ordered N-tuples ($N=n_1+\cdots+n_k$) of

integers x_{ij} satisfying

$$x_{ij} \geq r_i \, , \; i = 1,\ldots,k \quad \text{and} \quad \sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} = m.$$

b) They satisfy the following recurrence relations

$$S(m+1;\underline{n},\underline{r}) = N \, S(m;\underline{n},\underline{r}) + \sum_{j=1}^{k} \binom{m}{r_i-1} S(m-r_i+1;\underline{n}-\underline{e}_i,\underline{r}) \qquad (4.3)$$

$$S(m;\underline{n},\underline{r}+\underline{e}_i) = \sum_{j=0}^{n_i} (-1)^j \frac{(m)_{jr_i}}{j!(r_i!)^j} S(m-jr_i;\underline{n}-j\underline{e}_i,\underline{r}) \qquad (4.4)$$

with initial conditions

$$S(0;\underline{0},\underline{r}) = 1, \; S(0;\underline{n},\underline{r}) = 0 \text{ whenever } \sum r_i n_i > 0 \text{ and } S(m;\underline{n},\underline{r})=0 \text{ if } m<\underline{r}'\underline{n}. \qquad (4.5)$$

These follow from the difference-differential equation

$$\frac{d}{dt} f_{\underline{n}}(t;\underline{r}) = N f_{\underline{n}}(t,\underline{r}) + \sum_{i=1}^{k} f_{\underline{n}-\underline{e}_i}(t;\underline{r}) \, t^{r_i-1}/(r_i-1)! \qquad (4.6)$$

It can be easily seen that the representation (4.2) provides the following

combinatorial interpretation in terms of occupancy numbers.

Proposition 4.1. The number of ways of placing m distinguishable balls into
$N = n_1+\ldots+n_k$ cells so that each cell of the i-th group of n_i cells contains
at least r_i balls for $i = 1,\ldots,k$ is equal to $n_1!\ldots r_k!S(m;\underline{n},\underline{r})$ if the

N cells are distinguishable, and is equal to $S(m;\underset{\sim}{n},\underset{\sim}{r})$ if only cells belonging to different groups are distinguishable (and cells in the same group are alike).

It is easily concluded from Proposition 4.1, or from (4.3) - (4.5), that the numbers $S(m;\underset{\sim}{n},\underset{\sim}{r})$ are non-negative integers.

5. RATIOS OF MULTIPARAMETER STIRLING NUMBERS OF THE SECOND KIND.

We define, as ratio of multiparameter Stirling numbers of the second kind with respect to argument m, the function

$$R_2(m;\underset{\sim}{n},\underset{\sim}{r}) = \frac{S(m+1;\underset{\sim}{n},\underset{\sim}{r})}{S(m;\underset{\sim}{n},\underset{\sim}{r})} \tag{5.1}$$

Working as for Proposition 3.1, we obtain

Proposition 5.1. A recurrence relation for the ratio $R_2(m;\underset{\sim}{n},\underset{\sim}{r})$, independent of the multiparameter Stirling numbers of the second kind, is given by

$$R_2(m;\underset{\sim}{n},\underset{\sim}{r}) - N = \frac{\sum_{j=1}^{k} \frac{\binom{m}{r_j-1} r_j! \, n_j}{(\underset{\sim}{r}'\underset{\sim}{n})_{r_j}} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_2(m-r_j+1-i, \underset{\sim}{n}-e_j, \underset{\sim}{r})}{\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_2(m-i,\underset{\sim}{n},\underset{\sim}{r})} \tag{5.2}$$

for $n > 1$ and $m > r'n$ with

$$R_2(m,1,r) = k \tag{5.3}$$

and

$$R_2(\underset{\sim}{r}'\underset{\sim}{n};\underset{\sim}{n},\underset{\sim}{r}) = (\underset{\sim}{r}'\underset{\sim}{n}+1) \prod_{i=1}^{k} (r_i!)^{n_i} \sum_{j=1}^{k} \frac{n_j}{(r_j!)^{n_j-1} (r_j+1)! \prod_{\substack{i=1 \\ i \neq j}}^{k} (r_i!)^{n_i}} \tag{5.4}$$

The special case $k = 1$ yields

Proposition 5.2. A recurrence relation for the ratio $R_2(m,n,r)$,

independent of the generalized Stirling numbers of the second kind, is given by

$$R_2(m,n,r)-n = \frac{\frac{n\binom{m}{r-1}r!}{(rn)_r} \prod\limits_{i=1}^{m+1-rn} R_2(m-r+1-i,n-1,r)}{\prod\limits_{i=1}^{m-rn} R_2(m-i,n,r)} \tag{5.5}$$

for $n > 1$ and $m > rn$, with

$$R_2(m,1,r) = 1 \tag{5.6}$$

and

$$R_2(rn,n,r) = n(rn+1)/(r+1). \tag{5.7}$$

Also for $k = 1$, $r = 1$ we obtain

Proposition 5.3. A recurrence relation for the ratio $R_2(m,n)$, independent of the usual Stirling numbers of the second kind, is given by

$$R_2(m,n)-n = \frac{\prod\limits_{i=1}^{m+1-n} R_2(m-i,n-1)}{\prod\limits_{i=1}^{m-n} R_2(m-i,n)} \tag{5.8}$$

for $n > 1$ and $m > n$, with

$$R_2(m,1) = 1 \tag{5.9}$$

and

$$R_2(n,n) = n(n+1)/2. \tag{5.10}$$

Proposition 5.4. An altenative recurrence relation for the ratio $R_2(m,n,r)$ is given by

$$R_2(m,n,r)-n = \frac{m}{m-r+1} \frac{\left[R_2(m-1,n,r)-n\right] R_2(m-r,n-1,r)}{R_2(m-1,n,r)} \tag{5.11}$$

for $n > 1$ and $m > rn$.

Applying Proposition 5.4 with $r = 1$ gives

Proposition 5.5. An alternative recurrence relation for the ratio $R_2(m,n)$, is given by

$$R_2(m,n)-n = \frac{\left[R_2(m-1,n)-n\right] R_2(m-1,n-1)}{R_2(m-1,n)} \tag{5.12}$$

for $n > 1$ and $m > n$.

The last relation was also derived by Berg (1975).

6. MULTIPARAMETER C-NUMBERS.

The multiparameter C-numbers, $C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})$, are defined by their egf

$$\varphi_{\underset{\sim}{n}}(t;\underset{\sim}{s},\underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}) \frac{t^m}{m!} = \prod_{i=1}^{k} \frac{1}{n_i!}\left[(1+t)^{s_i} - \sum_{j=0}^{r_i-1}\binom{s_i}{j} t^j\right]^{n_i} \quad (6.1)$$

where the $s_i \neq 0$ $\ i = 1,\ldots,k$ are any real numbers.

Taking $k = 1$ gives the generalized C-numbers, Charalambides (1974a), and $k = 1$, $r_1 = 1$ defines the simple C-numbers, Cacoullos and Charalambides (1975), Charalambides (1977).

The following properties of $C(m,\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})$ are easily verified.

a) They have the representation.

$$C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}) = \frac{m!}{n_1!\ldots n_k!} \sum_{m} \prod_{i=1}^{k} \prod_{j=1}^{n_i} \binom{s_i}{x_{ij}} \quad (6.2)$$

where the summation extends over all ordered N-tuples $(N=n_1+\cdots+n_k)$ of integers x_{ij} satisfying

$$x_{ij} \geq r_i \quad i = 1,\ldots,k \quad \text{and} \quad \sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} = m .$$

b) They satisfy the following recurrence relations,

$$C(m+1;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}) = (\underset{\sim}{s}'\underset{\sim}{n}-m) C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}) + \sum_{i=1}^{k}\binom{m}{r_i-1}(s_i)_{r_i} C(m-r_i+1;\underset{\sim}{n}-\underset{\sim}{e}_i,\underset{\sim}{s},\underset{\sim}{r}), \quad (6.3)$$

$$C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}+\underset{\sim}{e}_i) = \sum_{j=0}^{n_i} (-1)^j \frac{(m)_{jr_i}}{j!} \binom{s_i}{r_i}^j C(m-jr_i;\underset{\sim}{n}-j\underset{\sim}{e}_i,\underset{\sim}{s},\underset{\sim}{r}), \quad (6.4)$$

with initial conditions

$C(0;\underset{\sim}{0},\underset{\sim}{s},\underset{\sim}{r}) = 1$, $C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}) = 0$ when $m < \underset{\sim}{r}'\underset{\sim}{n}$. They are obtained from the difference-differential equation

$$(1+t)\frac{d}{dt}\varphi_{\underset{\sim}{n}}(t;\underset{\sim}{s},\underset{\sim}{r}) = \underset{\sim}{s}'\underset{\sim}{n}\,\varphi_{\underset{\sim}{n}}(t;\underset{\sim}{s},\underset{\sim}{r}) + \sum_{i=1}^{k}\frac{(s_i)_{r_i}}{(r_i-1)!} t^{r_i-1} \varphi_{\underset{\sim}{n}-\underset{\sim}{e}_i}(t;\underset{\sim}{s},\underset{\sim}{r}). \quad (6.5)$$

The representation (6.2) leads to the following interpretation of the $C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})$-numbers in the framework of coupon-collecting problems.

Consider an urn containing k groups (sets) of distinguishable balls; the ith group consists of $s_i n_i$ balls and is divided into equal subgroups (subsets) of s_i balls each bearing the numbers $1,\ldots,n_i$; moreover, suppose that the balls of the k groups are distinguished by different colours so

that each ball in the urn is distinguished by its colour and number. Now it is easily seen from (6.2) that

Proposition 6.1. The number of ways of selecting m balls out of an urn with $\underset{\sim}{s}'\underset{\sim}{n} = \sum_{i=1}^{k} s_i n_i$ distinguishable balls, divided into k groups by colour and number as above into n_i subsets of size s_i within the i-th subgroup, so that each number $1,\ldots,n_i$ of the i-th subgroup (colour) appears at least r_i times is equal to

$$\frac{n_1! \ldots n_k!}{m!} \, C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}). \tag{6.6}$$

Here it was assumed that s_i is a positive integer. If s_i is a negative integer, say, $s_i = -s_i^*$, then

$$\prod_{i=1}^{k} \prod_{j=1}^{n_i} \binom{s_i}{x_{ij}} = \prod_{i=1}^{k} \prod_{j=1}^{n_i} \binom{-s_i^*}{x_{ij}} = \prod_{i=1}^{k} \prod_{j=1}^{n_i} (-1)^{x_{ij}} \binom{s_i^* + x_{ij} - 1}{x_{ij}} \tag{6.7}$$

and from (6.2) it can be concluded that the sign of $C(m,\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})$ is the same as $(-1)^m$. Furthermore, we may deduce

Proposition 6.2. The number of ways of distributing m ($m \geq \underset{\sim}{r}'\underset{\sim}{n}$) non-distinguishable balls into $\underset{\sim}{s}^*\,\underset{\sim}{n}$ cells, divided into k groups of cells with $s_i n_i$ cells in the i-th group and n_i subgroups each of s_i cells in the i-th group, so that each subgroup of the i-th group contains at least r_i balls is equal to

$$\frac{n_1! \ldots \ldots n_k!}{m!} \, \left| C(m;\underset{\sim}{n},-\underset{\sim}{s}^*,\underset{\sim}{r}) \right| . \tag{6.8}$$

As an indication of the applicability of the multiparameter C-numbers in occupancy problems, we refer to a problem posed by Sobel et. al.(1977),p.42.

Signless multiparameter C-numbers. From the basic recurrence relation (6.3) or from the last two propositions, we conclude that

(i) for $s_i > 0$ integer, the numbers $C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})$ are non-negative integers; they are positive for $\underset{\sim}{r}'\underset{\sim}{n} \leq m \leq \underset{\sim}{s}'\underset{\sim}{n}$; otherwise zero.

(ii) for $s_i < 0$ integer, the numbers $C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})$ are integers having the sign of $(-1)^m$.

Thus, as in the case of the Stirling numbers of the first kind, Riordan (1958),
the positive numbers

$$|C(m;\underset{\sim}{n},-\underset{\sim}{s}^*,\underset{\sim}{r})| = (-1)^m C(m;\underset{\sim}{n},-\underset{\sim}{s}^*,\underset{\sim}{r}) \tag{6.9}$$

will be called signless multiparameter C-numbers.

It can be easily verified that

Proposition 6.3. The egf of the signless multiparameter C-numbers $|C(m;\underset{\sim}{n},-\underset{\sim}{s},\underset{\sim}{r})|$,
$s_i > 0$ $i = 1,\ldots,k$ is given by

$$\varphi_{\underset{\sim}{n}}^*(t;-\underset{\sim}{s},\underset{\sim}{r}) = \prod_{i=1}^{k} \frac{1}{n_i!} \left[(1-t)^{-s_i} - \sum_{j=0}^{r_i-1} (-1)^j \binom{-s_i}{j} t^j \right]^{n_i}. \tag{6.10}$$

Remark. It should be observed that this is exactly the egf required for the
treatment of the mvue problem in the negative binomial case when the prob-
ability function of the i-th sample is

$$P(X = x_{ij}) = \frac{1}{g(\theta,r_i)} \binom{s_i+x_{ij}-1}{x_{ij}} \theta^{x_{ij}}(1-\theta)^{s_i} = (-1)^{x_{ij}} \binom{-s_i}{x_{ij}} \theta^{x_{ij}}(1-\theta)^{s_i} \tag{6.11}$$

with

$$g(\theta,r_i) = (1-\theta)^{-s_i} - \sum_{j=0}^{r_i-1} (-1)^j \binom{-s_i}{j} \theta^j, \quad i = 1,\ldots,k. \tag{6.12}$$

7. RATIOS OF MULTIPARAMETER C-NUMBERS.

We define, as ratio of multiparameter C-numbers with respect to argument
m, the function

$$R_3(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}) = \frac{C(m+1;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})}{C(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})}. \tag{7.1}$$

Proposition 7.1. A recurrence relation for the ratio $R_3(m,\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})$, independent
of the multiparameter C-numbers, is given by

$$R_3(m;\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})+m-\underset{\sim}{s}'\underset{\sim}{n} = \frac{\sum_{j=1}^{k} \frac{\binom{m}{r_j-1}(s_j)_{r_j} n_j}{(\underset{\sim}{r}'\underset{\sim}{n})_{r_j}\binom{s_j}{r_j}} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_3(m-r_j+1-i,\underset{\sim}{n}-\underset{\sim}{e}_j,\underset{\sim}{s},\underset{\sim}{r})}{\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_3(m-i,\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r})} \tag{7.2}$$

for $\underset{\sim}{n} > \underset{\sim}{1}$ and $m > \underset{\sim}{r}'\underset{\sim}{n}$, with

$$R_3(m,1,s,r) = r-m \tag{7.3}$$

$$R_3(\underset{\sim}{r}'\underset{\sim}{n};\underset{\sim}{n},\underset{\sim}{s},\underset{\sim}{r}) = \frac{(\underset{\sim}{r}'\underset{\sim}{n}+1)}{\prod_{i=1}^{k}\binom{s_i}{r_j}^{n_j}} \sum_{j=1}^{k} n_j \binom{s_j}{r_j+1}\binom{s_j}{r_j}^{n_j-1} \prod_{\substack{j=1\\i\neq j}}^{k}\binom{s_j}{r_j}^{n_j} \tag{7.4}$$

Proposition 7.2. A recurrence relation for the ratio $R_3(m,n,s,r)$, independent of the generalized C-numbers (case $k=1$), is given by

$$R_3(m,n,s,r)+m-sn = \frac{\dbinom{m}{r-1}\dfrac{(s)_r}{(rn)_r}\dfrac{n}{\dbinom{s}{r}}\displaystyle\prod_{i=1}^{m+1-rn} R_3(m-r+1-i,n-1,s,r)}{\displaystyle\prod_{i=1}^{m-rn} R_3(m-i,n,s,r)} \tag{7.5}$$

for $n > 1$ and $m > rn$, with

$$R_3(m,1,s,r) = s-m \tag{7.6}$$

and

$$R_3(rn,n,s,r) = n(rn+1)(s-r)/(r+1) \tag{7.7}$$

Proposistion 7.3. A recurrence relation for the ratio $R_3(m,n,s)$, independent of the usual C-numbers (case $r=1$), is given by

$$R_3(m,n,s)+m-sn = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_3(m-i,n-1,s)}{\displaystyle\prod_{i=1}^{m-n} R_3(m-i,n,s)} \tag{7.8}$$

for $n > 1$ and $m > n$, with

$$R_3(m,1,s) = s-m \tag{7.9}$$

and

$$R_3(n,n,s) = (s-1)n(n+1)/2 . \tag{7.10}$$

Proposition 7.4. An alternative recurrence relation for the ratio $R_3(m,n,s,r)$, is given by

$$R_3(m,n,s,r)+m-sn = \frac{m}{m-r+1}\frac{\left[R_3(m-1,n,s,r)+m-sn-1\right] R_3(m-r,n-1,s,r)}{R_3(m-1,n,s,r)} \tag{7.11}$$

for $n > 1$ and $m > rn$.

Proposition 7.5. An alternative recurrence relation for the ratio $R_3(m,n,s)$, is given by

$$R_3(m,n,s)+m-sn = \frac{\left[R_3(m-1,n,s)+m-sn-1\right] R_3(m-1,n-1,s)}{R_3(m-1,n,s)} \tag{7.12}$$

for $n > 1$ and $m > n$.

8. RELATIONS BETWEEN THE STIRLING AND C-NUMBERS.

It was observed in Cacoullos and Charalambides (1975), that

$$\lim_{s \to \pm\infty} s^{-m} C(m,n,s) = S(m,n) \tag{8.1}$$

that is, the C-numbers can be approximated by the Stirling numbers of the second kind for large s, a fact which reflects the corresponding well known convergence of the binomial to the Poisson (s → ∞ , p → 0 , i.e. θ = p/q → 0 and hence sp or sθ converges to the Poisson parameter λ). The above property extends to the case of multiparameter Stirling numbers of the second kind and multiparameter C-numbers, namely,

$$\lim_{s_i \to \pm\infty} s_i^{-m} C(m;n,s,r) = S(m;n,r), \quad i = 1,\dots,k. \tag{8.2}$$

This can easily be verified by using the corresponding representation (4.2) and (6.2) of these numbers and noting that

$$\lim_{s \to \pm\infty} s^{-k} \binom{s}{k} = 1/k!. \tag{8.3}$$

A relation between the signless multiparameter Stirling numbers of the first kind and the multiparameter C-numbers reflects the limiting relation between the negative binomial and the logarithmic series distributions:

$$\lim_{s_i \to 0} s_i^{-N} |C(m;n,-s,r)| = |s(m;n,r)| , \quad N = n_1 + \dots + n_k. \tag{8.4}$$

This can be seen, eg, by showing that the egf of the $s_i^{-N}|C(m,n,-s,r)|$-numbers converges to the egf of the $|s(m;n,r)|$-numbers, that is,

$$\lim_{s_i \to 0} \frac{1}{s_i^N} \prod_{i=1}^{k} \frac{1}{n_i!} \left[(1-t)^{-s_i} - \sum_{j=0}^{r_i-1} (-1)^j \binom{-s_i}{j} t^j \right]^{n_i} = \prod_{i=1}^{k} \frac{1}{n_i!} \left[-\log(1-t) - \sum_{j=1}^{r_i-1} \frac{t^j}{j} \right]^{n_i}. \tag{8.5}$$

For this note that

$$\frac{1}{s} (-1)^j \binom{-s}{j} t^j \xrightarrow[s \to 0]{} \frac{t^j}{j} . \tag{8.6}$$

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APPENDIX B

CHARACTERIZATIONS OF DISCRETE DISTRIBUTIONS BY A CONDITIONAL DISTRIBUTION AND A REGRESSION FUNCTION*

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Key Words and Phrases: Characterizations,discrete mixtures, identifiability, regression.

Abstract

The bivariate distribution of (X,Y), where X and Y are non-negative integer-valued random variables, is characterized by the conditional distribution of Y given X and a consistent regression function of X on Y. This is achieved when the conditional distribution is one of the distributions:a) binomial, Poisson, Pascal or b) a right translation of these. In a) the conditional distribution of Y is an x-fold convolution of another random variable independent of X so that Y is a generalized distribution. A main feature of these characterizations is that their proof does not depend on the specific form of the regression function. Moreover, it is shown that the characterizations hold if the regression function is replaced by any higher-order conditional moment. It is also indicated how these results can be used for goodness-of-fit purposes.

1. INTRODUCTION

Here we are concerned with characterizing the distribution of non-negative integer-valued random variables (r.v.) X and Y in terms of the conditional distribution of Y given X and the regression function $E(X|Y)$ of X on Y. Several papers have appeared in this direction. Korwar (1975) considered a conditional binomial or Pascal distribution combined with linear regression and characterized the Poisson, binomial and negative binomial distributions in the former case and the geometric in the latter case. Dahiya and Korwar (1977) extended these characterizations to bivariate X and Y under conditional distributions which are independent binomials or Pascal and linear regression. Khatri (1978a),(1978b),using a slightly more general approach gave similar results for the multivariate case. A case of non-linear regression was treated by Xekalaki (1980) in characterizing the bivariate Poisson distribution.

In this paper more general and unifying results are obtained by by-passing the unnecessary details involved in obtaining a specific characterization under a specific regression function. This is achieved by appealing to the unicity of a solution of a first-order difference equation. Specifically, it is shown that certain conditional distributions along with the regression functions determine uniquely the distributions of X and Y, hence also of (X,Y). This fact can be used to generate a wide spectrum of distributions characterized under these conditions. For example, given a conditional binomial distribution (of Y on X), we may choose an arbitrary given distribution for X, which in turn gives a specific regression function; thus (see Section 4), in addition to the distributions mentioned earlier, characterizations were obtained for the logarithmic distribution and several generalized distributions, e.g., Neyman, Poisson binomial, binomial Poisson, logarithmic binomial etc.

Finally, it is shown that the distribution of (X, Y) can be similarly characterized if we replace $E(X|Y)$ by the kth conditional moment $E[X^k|Y]$ for some value of k. It should also be added that the present approach goes through in the multivariate case, which will be treated in a subsequent paper.

2. SOME PRELIMINARIES

We shall make use of the following easily verified combinatorial identities:

$$x\binom{x}{y} = (y+1)\binom{x}{y+1} + y\binom{x}{y} , \tag{2.1}$$

$$x^k\binom{x}{y} = A_k(y)\binom{x}{y+k} + A_{k-1}\binom{x}{y+k-1} + \cdots + A_0(y)\binom{x}{y} , \tag{2.1'}$$

$$x\binom{y-1}{x-1} = y\binom{y-1}{x-1} - (y-1)\binom{y-2}{x-1} , \tag{2.2}$$

$$x^k\binom{y-1}{x-1} = B_k(y)\binom{y-k-1}{x-1} + B_{k-1}\binom{y-k-2}{x-1} + \cdots + B_0\binom{y-1}{x-1} , \tag{2.2'}$$

$$x\binom{x+y-1}{y} = (y+1)\binom{x+y}{y+1} - y\binom{x+y-1}{y} , \tag{2.3}$$

$$x^k\binom{x+y-1}{y} = C_k(y)\binom{x+y+k-1}{y+k} + C_{k-1}(y)\binom{x+y+k-2}{y+k-1} + \cdots + C_0(y)\binom{x+y-1}{y} \tag{2.3'}$$

where the $A_i(y), B_i(y), C_i(y)$ denote polynomials in y of degree k. In $\binom{m}{r}$ m and r are integers and $r \geqslant 0$.

Moreover, a great role in the sequel is played by

Theorem 2.1 (Goldberg 1958, p.61). The linear difference equation of order n

$$f_0(k)y_{k+n} + f_1(k)y_{k+n-1} + \cdots + f_{n-1}(k)y_{k+1} + f_n(k)y_k = g(k)$$

over a set of consecutive integer values of k has one, and only one, solution for which values at n consecutive k-values are arbitrarily prescribed.

3. THE MAIN CHARACTERIZATION THEOREMS

Certain forms of the conditional distribution $p(y|x)$ of Y given $X=x$ together with the regression function $m(y) = E[X|Y=y]$ of X on Y determine the distributions of X and Y. It is to be observed from the outset that in the main case considered here, namely, when $p(y|x)$ is an x-fold convolution of a non-negative integer-valued r.v. with probability generating function (p.g.f.) $h_0(v)$, it is sufficient to determine the distribution of either X or Y. This is so because the p.g.f.'s $g(u)$ of X, $h(v)$ of Y and $G(u,v)$ of (X,Y) Cacoullos and Papageorgiou (1981b) satisfy (3.1) (see also (3.6a)-(3.10a);equivalently, because of the identifiability of the mixtures defined by (3.6)-(3.10), with mixing variable (parameter) x (see e.g., Teicher 1961).

$$h(v) = g(h_0(v)), \quad G(u,v) = g(uh_0(v)) \qquad (3.1)$$

Here, we find first the probability function $p(y)$ of Y.

A similar remark applies to the case in which $p(y|x)$ determines a shift to the right by x of a non-negative integer-valued r.v. with p.g.f. $h^*(v)$, so that denoting by $h_x(v)$ the conditional p.g.f. of Y given $X=x$,

$$h_x(v) = v^x h^*(v), \quad h(v) = h^*(v)g(v). \qquad (3.2)$$

A characteristic of the following characterizations is that their derivation is independent of the actual form of the regression function $m(y)$; this was confined to be linear in the relevant statistical literature. The simplification is achieved by employing a basic theorem about the uniqueness of a solution of linear difference equation (Theorem 2.1). The difference equation governing $p(y)$ in each case is obtained by using Lemma 3.1. Let $p(y|x)$ be such that there exist p.g.f.'s h_0, h^* independent of x and functions c_i of y (only) so that

$$h_x(v) = [h_0(v)]^x \quad \text{or} \quad h_x(v) = v^x h^*(v) \quad \text{for } x=0,1,2,\dots \qquad (3.3)$$

$$xp(y|x) = c_1(y)p(y+1|x)+c_0(y)p(y|x)+c_{-1}(y)p(y-1|x) \qquad (3.4)$$

Then $p(y|x)$ and $m(y)$ characterize the distribution of Y; hence of X and (X,Y).

Proof. We have

$$m(y) = \sum_{x=0}^{\infty} xP[X=x|Y=y] = \sum_{x=0}^{\infty} x\frac{p(y|x)}{p(y)} P[X=x]$$

which by (3.4) can be written as

$$m(y) = \left[c_1(y)p(y+1)+c_0(y)p(y)+c_{-1}(y)p(y-1)\right]/p(y) \qquad (3.5)$$

Since this is a linear second-order difference equation in $p(y)$ the assertion follows by Theorem 2.1 and in virtue of (3.1) and (3.2).

Note: In the applications of the Lemma, either $c_1(y)$ or $c_{-1}(y)$ is zero so that (3.5) reduces to a first-order difference equation. Now we state the main theorems.

Theorem 3.1. Let the conditional distribution $p(y|x)$ be one of the distributions (3.6)-(3.10). Let $m(y) = E[X|Y=y]$ be an arbitrary function of y consistent with $p(y|x)$. Then $p(y|x)$ and $m(y)$ together determine the distributions of X,Y and (X,Y).

$$p(y|x) = \binom{x}{y}p^y q^{x-y} , \quad y=0,\ldots,x, \quad x=0,1,\ldots \quad (q=1-p) \qquad (3.6)$$

$$p(y|x) = \binom{y-1}{x-1}p^x q^{y-x} , \quad y=x,x+1,\ldots, \quad x=1,2,\ldots \qquad (3.7)$$

$$p(y|x) = \binom{x+y-1}{y}\left(\frac{P}{Q}\right)^y (1-\frac{P}{Q})^x, \quad y=0,1,\ldots, \quad x=1,\ldots \quad (P=Q-1) \qquad (3.8)$$

$$p(y|x) = e^{-\lambda x} \frac{(\lambda x)^y}{y!} , \quad y=0,1,\ldots, \quad x=0,1,\ldots \qquad (3.9)$$

$$p(y|x) = \binom{nx}{y}p^y q^{nx-y} \quad y=0,1,\ldots,nx \quad x=0,1,\ldots \qquad (3.10)$$

Proof. By Lemma 3.1, it is enough to show that for some p.g.f. $h_0(v)$

$$h_x(v) = \left[h_0(v)\right]^x$$

and, moreover, that $p(y|x)$ satifies (3.4).

Indeed, the $h_0(v)$ corresponding to the distributions (3.6)-(3.10) and the respective p.g.f.'s of X and Y (see (3.1)) are as follows:

$$h_0(v) = pv+q, \qquad g(v) = h(\frac{v-q}{p}) \tag{3.6a}$$

$$h_0(v) = \frac{pv}{1-qv}, \qquad g(v) = h(\frac{v}{p+qv}) \tag{3.7a}$$

$$h_0(v) = (Q-Pv)^{-1}, g(v) = h(\frac{Qv-1}{Pv}) \tag{3.8a}$$

$$h_0(v) = e^{\lambda(v-1)}, \; g(v) = h(\frac{\log v}{\lambda}+1) \tag{3.9a}$$

$$h_0(v) = (pv+q)^n, \; g(v) = h(\frac{v^{1/n}-q}{p}) \tag{3.10a}$$

As regards (3.4), by using (2.1) for (3.6) and (3.10), (2.2) for (3.7) and (2.3) for (3.8), we obtain the respective difference equations (recurrences) of the first order :

$$m(y) = \frac{q}{p}(y+1)\frac{p(y+1)}{p(y)} +y \qquad\qquad y=0,1,\ldots \tag{3.6b}$$

$$m(y) = y-q(y-1)\frac{p(y-1)}{p(y)} \qquad\qquad y=1,2,\ldots \tag{3.7b}$$

$$m(y) = (y+1)\frac{Q}{P}\frac{p(y+1)}{p(y)} - y \qquad\qquad y=0,1,\ldots \tag{3.8b}$$

$$m(y) = \frac{y+1}{\lambda}\frac{p(y+1)}{p(y)} \qquad\qquad y=0,1,\ldots \tag{3.9b}$$

$$m(y) = \frac{q}{np}(y+1)\frac{p(y+1)}{p(y)} + \frac{y}{n} \qquad\qquad y=0,1,\ldots \tag{3.10b}$$

Hence the proof of the theorem is complete.

The explicit solutions of the above difference equations (3.6b)-(3.10b) are respectively :

$$p(y) = p(0)\left(\frac{p}{q}\right)^y \prod_{k=0}^{y-1} \frac{1}{k+1}(m(k)-k), \quad y=1,2,\ldots \tag{3.6c}$$

$$p(y) = p(1)q^{y-1} \prod_{k=1}^{y} \frac{1-k}{m(k)-k}, \qquad y=2,3,\ldots \tag{3.7c}$$

$$p(y) = p(0)\left(\frac{P}{Q}\right)^y \prod_{k=0}^{y-1} \frac{1}{k+1}(m(k)+k), \quad y=1,2,\ldots \tag{3.8c}$$

$$p(y) = p(0)\lambda^y \prod_{k=0}^{y-1} \frac{m(k)}{k+1} , \qquad\qquad y=1,2,\dots \qquad (3.9c)$$

$$p(y) = p(0)\left(\frac{p}{q}\right)^y \prod_{k=0}^{y-1} \frac{1}{k+1} (nm(k)-k), \qquad y=1,2,\dots \qquad (3.10c)$$

where $p(0)$ or $p(1)$ are determined from the condition $\sum_y p(y)=1$.

Notice that (3.6) is a special case of (3.10) with $n=1$. Also that (3.7) and (3.8) correspond to the two versions of a Pascal r.v., which denotes the number of failures in (3.8) and the number of trials up to the xth success in (3.7).

Next we consider the case described by (3.2). We prove

<u>Theorem 3.2</u> Let $p(y|x)$ be one of the distributions (3.11)-(3.13). Let $m(y)$ be consistent with $p(y|x)$. Then $p(y|x)$ and $m(y)$ together determine the distributions of X,Y and (X,Y).

$$p(y|x) = \binom{n}{y-x} p^{y-x} q^{n-y+x} \qquad x \leqslant y \leqslant x+n, \qquad (3.11)$$

$$p(y|x) = e^{-\lambda} \frac{\lambda^{y-x}}{(y-x)!} \qquad y \geqslant x, \quad x=0,1,\dots \qquad (3.12)$$

$$p(y|x) = \binom{N+y-x-1}{N-1}\left(\frac{P}{Q}\right)^{y-x} (1-\frac{P}{Q})^N \qquad y \geqslant x \geqslant 0, \quad (P=Q-1) \qquad (3.13)$$

<u>Proof</u>. Apply Lemma 3.1 to (3.11)-(3.13) with h^* given, respectively, by

$$h^*(v) = (pv+q)^n \qquad\qquad (3.11a)$$

$$h^*(v) = e^{\lambda(v-1)} \qquad\qquad (3.12a)$$

$$h^*(v) = (Q-Pv)^{-N} \qquad\qquad (3.13a)$$

Now using the identity

$$x\binom{n}{y-x} = y\binom{n}{y-x} - (n-y+1)\binom{n}{y-x-1} - x\binom{n}{y-x-1}$$

we can write $m(y)$ for (3.11) in the form

$$m(y) = y-(n-y+1) \frac{p}{q} \frac{p(y-1)}{p(y)} + \frac{p}{q} \frac{p(y-1)}{p(y)} m(y-1).$$ (3.11b)

For (3.12) we easily find

$$m(y) = y-\lambda \frac{p(y-1)}{p(y)}$$ (3.12b)

and for (3.13), making use of the identity

$$x\binom{N+y-x-1}{N-1} = y\binom{N+y-x-1}{N-1} - (N+y-1)\binom{N-y-x-2}{N-1} + x\binom{N+y-x-2}{N-1},$$

we have

$$m(y) = y-(N+y-1) \frac{P}{Q} \frac{p(y-1)}{p(y)} + \frac{P}{Q} \frac{p(y-1)}{p(y)} m(y-1)$$ (3.13b)

Here again $p(y)$ is obtained as the solution of the corresponding first-order difference equation. In point of fact, we have

$$p(y) = p(0)\left(\frac{p}{q}\right)^y \prod_{k=1}^{y} \frac{n-k+1+m(k-1)}{k-m(k)},$$ (3.11c)

$$p(y) = p(0) \lambda^y \prod_{k=1}^{y} \frac{1}{k-m(k)},$$ (3.12c)

$$p(y) = p(0)\left(\frac{P}{Q}\right)^y \prod_{k=1}^{y} \frac{m(k-1)-(N+k-1)}{m(k)-k}$$ (3.13c)

where $p(0)$ is also determined since $\sum\limits_{y=0}^{\infty} p(y)=1$.

Finally, we give another set of characterizations under the assumptions of Theorem 3.1, i.e. (3.6)-(3.10), replacing $m(y) = E[X|Y=y]$ by a higher conditional moment $m_k(y) = E[X^k|Y=y]$ for a $k>1$.

Theorem 3.3. For a $k>1$, let $m_k(y)$ be given function of y which is consistent with one of the conditional distributions (3.6)-(3.10). Then $p(y|x)$ and $m_k(y)$ together determine $p(y)$ up to $k-1$ arbitrary probabilities, say, $p(1),\ldots,p(k-1)$.

Proof. By using the generalized identities $(2.1)'-(2.3)'$ and working as in Theorem 3.1, we obtain a k-th order difference equation in $p(y)$.

Hence the assertion.

Note. The undetermined $p(1),\ldots,p(k-1)$ may be obtained by using e.g. as initial conditions the values of $m(y)$ for $y=1,\ldots,k-1$, since we always have the condition $\sum\limits_{y=0}^{\infty} p(y)=1$, thus determining $p(0)$.

4. SOME APPLICATIONS-COROLLARIES OF THEOREM 3.1

It was already pointed out that in characterizing the distribution of X,Y and (X,Y) in terms of the conditional distribution $p(y|x)$ of Y given $X=x$ and $m(y)=E[X|Y=y]$, the regression function $m(y)$ must originate from some non-negative integer-valued r.v. X and be consistent with $p(y|x)$. This is the spirit of the following characterizations-applications of Theorem 3.1. In the applications of Theorem 3.1, it should be observed that, for a specific r.v. X,Y has the generalized (compound) distribution of X by another r.v. Z_i (denoted by XvZ_i), since Y has the representation

$$Y = Z_1+\cdots+Z_X$$

where the Z_i are i.i.d.,with p.g.f. $h_0(v)$, independent of X.

The following characterizations, in addition to the usual discrete distributions, cover some more involved generalized discrete distributions, of which the more interesting ones are presented here.

Proposition 4.1. Suppose (3.6) holds. Then we have:

(a) For some $\lambda>0$

$$m(y) = y+\lambda q \quad , \quad y=0,1,2,\ldots,$$

iff X is Poisson (λ); then Y is also Poisson (λp) and (X,Y) is a bivariate Poisson-Bernoulli model, studied by Leiter and Hamdan (1973) and Cacoullos and Papageorgiou (1980).

(b) For some $0<p'=1-q'<1$ and some integer $n>0$

$$m(y) = \frac{q'}{q'+p'q} y+n \frac{p'q}{q'+p'q} , \qquad 0\le y\le n.$$

iff X is binomial (n,p'); then Y is also a binomial (n,pp') and (X,Y) is a special case of a bivariate binomial distribution.

(c) For some $N>0$, $P=Q-1>0$

$$m(y) = \frac{Q}{Q-Pq}y + \frac{NPq}{Q-Pq} \qquad y=0,1,\ldots$$

iff X is negative binomial $NB(N,P)$; then Y is also $NB(N,Pp)$ and (X,Y) is a bivariate negative binomial-Bernoulli model, studied by Cacoullos and Papageorgiou (1981a).

(d) For some $0<\theta<1$

$$m(y) = \frac{1}{1-\theta q}\, y \qquad y=1,2,\ldots$$

$$m(0) = -\frac{1}{\log(1-\theta q)}\frac{\theta q}{1-\theta q}$$

iff X is logarithmic (θ); then Y follows a modified logarithmic (δ,θ'), with δ the probability of $Y=0$ and θ' its ordinary parameter, where

$$\delta = p(0) = \frac{\log(1-\theta q)}{\log(1-\theta)}, \quad \theta' = \frac{p\theta}{1-\theta q} \;;$$

(X,Y) is a bivariate logarithmic-Bernoulli model (cf. Cacoullos and Papageorgiou, (1981b).

Remark. Cases (a), (b) and (c), $m(y) = ay+b$, $y=0,1,\ldots$, as considered by Korwar (1975). Case (d), exhibiting linearity only for $y>0$, was missed by Korwar in view of the limitations of treating the problem in terms of a specified regression function. The present more general approach allows a variety of $m(y)$, including those which correspond to truncated versions of X. For example, if X is truncated on the left at r, $m(y)$ is linear only for $y\geq r$ under (3.6) and the corresponding characterizations hold with appropriate modifications for Y and (X,Y).

(e) For some positive λ and θ

$$m(y) = y + \frac{q\theta S_{y+1}(c_1)}{S_y(c_1)} \qquad y=0,1,\ldots$$

where $S_n(t)$ is a Stirling polynomial defined by Charalambides (1977),

$$S_n(t) = \sum_{k=0}^{n} S(n,k)t^k \ , \quad c_1 = \lambda e^{-\theta p} \ ,$$

and $S(n,k)$ denotes a Stirling number of the second kind,

iff X has a Neyman distribution, i.e., Poisson (λ) vPoisson (θ);

then Y has also a Neyman: Poisson (λ) vPoisson (θp) and (X,Y),

with pgf

$$G(u,v) = \exp\left[\lambda\{e^{\theta\left[u(q+pv)-1\right]} -1\}\right] \ ,$$

is a special case of a bivariate Neyman Type I distribution, Holgate (1966).

(f) For some $0 < p' = 1 - q' < 1$, $\lambda > 0$ and an integer $n > 0$

$$m(y) = \frac{p'q}{q'+p'q} \frac{C_{y+1,n}(c_2)}{C_{y,n}(c_2)} + y \ , \quad y=0,1,\ldots$$

$C_{y,r}$

where $C_{y,n}(t)$ is the polynomial, Charalambides (1977),

$$C_{y,n}(t) = \sum_{k=0}^{y} C(y,k,n)t^k \ , \qquad c_2 = \lambda(q'+p'q)^n \ ,$$

and $C(y,n,k)$ are the C-numbers, iff X is Poisson (λ) v binomial (n,p');

then Y is also a Poisson (λ) v binomial $(n,p'p)$.

It is worth noting that (X,Y) with p.g.f.

$$G(u,v) = \exp\left[\lambda\{\left[q'+p'u(q+pv)\right]^n -1\}\right]$$

is a special case of the bivariate Poisson binomial type I distribution
studied by Charalabmides and Papageorgiou (1981).

(g) For some $P=Q-1>0$, $\lambda>0$ and an $N>0$

$$m(y) = y - \frac{Pq}{Q-Pq} \frac{C_{y+1,-N}(c_3)}{C_{y,-N}(c_3)} \qquad y=0,1,2,\ldots$$

where

$$C_{y,-N}(t) = \sum_{k=0}^{y} C(y,k,-N)t^k, \qquad c_3 = \lambda(Q-Pq)^{-N},$$

iff X is Poisson (λ) v negative binomial (N,P); then, Y is also Poisson (λ) v negative binomial (N,Pp).

(h) For some $0<p'=1-q'<1$; $\lambda>0$ and an integer $n>0$

$$m(y) = \frac{\lambda q S_{y+1,n}(c_4)}{S_{y,n}(c_4)} + y \qquad y=0,1,2,\ldots$$

where

$$S_{y,n}(t) = \sum_{k=0}^{y} (n)_k S(y,k)t^k, \qquad c_4 = p'e^{-\lambda P}(q'+p'e^{-\lambda P})^{-1}$$

iff X is binomial (n,p') v Poisson (λ); then Y is also binomial (n,p') v Poisson (λp).

(i) For some $0<p'=1-q'<1, 0<p^*=1-q^*<1$ and positive integers n,n^*

$$m(y) = \frac{p^*q}{q^*+p^*q} \frac{C_{y+1,n,n^*}(c_5)}{C_{y,n,n^*}(c_5)} + y, \qquad 0\leq y\leq nn^*,$$

where

$$C_{y,n,n^*}(t) = \sum_{k=0}^{y} (n)_k C(y,k,n^*)t^k$$

and

$$c_5 = p'(q^*+p^*q)^{n^*}\{q'+p'(q^*+p^*q)^{n^*}\}^{-1},$$

iff X is a binomial (n,p') v binomial (n^*,p^*); then Y is also a binomial (n,p') v binomial (n^*,p^*p)

(j) For some $P=Q-1>0$, $0<p'=1-q'<1$ an integer $n>0$ and an $N>0$

$$m(y) = y - \frac{Pq}{Q-Pq} \frac{C_{y+1,n,-N}(c_6)}{C_{y,n,-N}(c_6)} \qquad y=0,1,2,\ldots$$

where

$$C_{y,n,-N}(t) = \sum_{k=0}^{y} (n)_k \, C(y,k,-N)t^k$$

and

$$c_6 = p'(Q-Pq)^{-N}\{q'+p'(Q-Pq)^{-N}\}^{-1} ,$$

iff X is binomial (n,p') v negative binomial (N,P); then Y is also a binomial (n,p') v negative binomial (N,Pp).

(k) For some $P=Q-1>0$, $\lambda>0$ and $N>0$

$$m(y) = \frac{\lambda q \, S_{y+1,-N}(c_7)}{S_{y,-N}(c_7)} + y \qquad y=0,1,2,\ldots$$

where

$$S_{y,-N}(-t) = \sum_{k=0}^{y} (-1)^k(-N)_k \, S(y,k)t^k$$

and

$$c_7 = Pe^{-\lambda P}(Q-Pe^{-\lambda P})^{-1} ,$$

iff X is negative binomial (N,P) v Poisson (λ); then Y is also a negative binomial (N,P) v Poisson (λp).

(ℓ) For some $0<p'=1-q'<1, P=Q-1>0$, $N>0$ and an integer $n>0$

$$m(y) = \frac{p'q}{q'+p'q} \, \frac{C_{y+1,-N,n}(-c_8)}{C_{y,-N,n}(-c_8)} + y \qquad y=0,1,2,\ldots$$

where

$$C_{y,-N,n}(-t) = \sum_{k=0}^{y} (-1)^k(-N)_k \, C(y,k,n)t^k$$

and

$$c_8 = P(q'+p'q)^n\{Q-P(q'+p'q)^n\}^{-1} ,$$

iff X is negative binomial (N,P) v binomial (n,p'); then Y is also a negative binomial (N,P) v binomial $(n,p'p)$.

(m) For some $P=Q-1>0$, $P^*=Q^*-1>0$ and $N>0$, $N^*>0$

$$m(y) = y - \frac{P^*q}{Q^*-P^*q} \, \frac{C_{y+1,-N,-N^*}(-c_9)}{C_{y,-N,-N^*}(-c_9)} \qquad y=0,1,2,\ldots$$

where

$$C_{y,-N,-N*}(-t) = \sum_{k=0}^{y} (-1)^k (-N)_k \, C(y,k,-N*) \, t^k$$

and

$$c_9 = P(Q*-P*q)^{-N*}\{Q-P(Q*-P*q)^{-N*}\}^{-1} \ ,$$

iff X is negative binomial (N,P) v negative binomial $(N*,P*)$; then Y is also negative binomial (N,P) v negative binomial $(N*,P*p)$

(n) For some $0<\theta<1$, $\lambda>0$

$$m(y) = \frac{\lambda q \, S*_{y+1}(c_{10})}{S*_y(c_{10})} + y \, , \qquad y=1,2,\dots \ ,$$

$$m(0) = \frac{-\lambda q S*_1(c_{10})}{\log(1-\theta e^{-\lambda P})} \ ,$$

where

$$S*_y(t) = \sum_{k=1}^{y} (k-1)! \, S(y,k) t^k$$

and

$$c_{10} = \theta e^{-\lambda P}\{1-\theta e^{-\lambda P}\}^{-1} \ ,$$

iff X is logarithmic (θ) v Poisson (λ); then Y is also a logarithmic (θ) v Poisson (λp).

(o) For some $0<\theta<1$, $0<p'=1-q'<1$ and an integer $n>0$

$$m(y) = \frac{p'q}{q'+p'q} \frac{C*_{y+1,n}(c_{11})}{C*_{y,n}(c_{11})} + y \qquad y=1,2,\dots$$

$$m(0) = \frac{-1}{\log\{1-\theta(q'+p'q)^n\}} \frac{p'q}{q'+p'q} \, C*_{1,n}(c_{11}) \ ,$$

where

$$C*_{y,n}(t) = \sum_{k=1}^{y} (k-1)! \, C(y,k,n) t^k$$

and

$$c_{11} = \theta(q'+p'q)^n\{1-\theta(q'+p'q)^n\}^{-1} \ ,$$

iff X is logarithmic (θ) v binomial (n,p'); then Y is also logarithmic (θ) v binomial $(n,p'p)$.

(p) For some $0<\theta<1$, $P=Q-1>0$ and $n>0$

$$m(y) = y - \frac{Pq}{Q-Pq} \frac{C^{*}_{y+1,-N}(c_{12})}{C^{*}_{y,-N}(c_{12})} \qquad y=1,2,\ldots$$

$$m(0) = \frac{1}{\log\{1-\theta(Q-Pq)^{-N}\}} \frac{Pq}{Q-Pq} C^{*}_{1,-N}(c_{12}) \ ,$$

where

$$C^{*}_{y,-N}(t) = \sum_{k=1}^{y} (k-1)! \ C(y,k,-N)t^{k}$$

and

$$c_{12} = \theta(Q-Pq)^{-N}\{1-\theta(Q-Pq)^{-N}\}^{-1} \ ,$$

iff X is logarithmic (θ) v negative binomial (N,P); then Y is also a logarithmic (θ) v negative binomial (N,Pp).

Proposition 4.2. Suppose (3.7) holds. Then we have:

(a) For some $0<p'=1-q'<1$

$$m(y) = \frac{qp}{q'p+q} y + \frac{q}{q'p+q} \qquad y=1,2,\ldots,$$

iff X is geometric (p'); then Y is also geometric $(p'p)$.

(b) For some $0<\theta<1$

$$m(y) = y - q \frac{(q+\theta p)^{y-1}-q^{y-1}}{(q+\theta p)^{y}-q^{y}} \ , \quad y=1,2,\ldots$$

iff X is logarithmic (θ); then Y is a logarithmic (θ) v geometric (p).

Proposition 4.3. Suppose (3.8) holds. Then we have :

(a) For some $P=Q-1>0$ and a $\lambda>0$

$$m(y) = - \frac{C_{y+1,-1}(c_{13})}{C_{y,-1}(c_{13})} - y \qquad y=0,1,2,\ldots,$$

where $C_{y,-1}(t)$ is defined under Proposition 4.1 (g) and

$$c_{13} = \lambda Q^{-1} ,$$

iff X is a Poisson (λ); then Y is a Poisson (λ) v geometric $(1/Q)$.

(b) For some $0<p=1-q<1$ and an integer $n>0$

$$m(y) = - \frac{C_{y+1,n,-1}(c_{14})}{C_{y,n,-1}(c_{14})} - y , \qquad y=0,1,2,\ldots,$$

where $C_{y,n,-1}(t)$ is defined under Proposition 4.1 (j) and

$$c_{14} = pQ^{-1}(q+pQ^{-1})^{-1} ,$$

iff X is a binomial (n,p); then Y is a binomial (n,p) v geometric $(1/Q)$.

(c) For some $P'=Q'-1>0$

$$m(y) = - \frac{C_{y+1,-N,-1}(-c_{15})}{C_{y,-N,-1}(-c_{15})} - y , \qquad y=0,1,2,\ldots,$$

where $C_{y,-N,-1}(-t)$ is defined under Proposition 4.1 (m) and

$$c_{15} = P'Q^{-1}\{Q'-P'Q^{-1}\}^{-1} ,$$

iff X is a negative binomial (N,P'); then Y is negative binomial (N,P') v geometric $(1/Q)$.

(d) For some $0<\theta<1$

$$m(y) = - \frac{C^{*}_{y+1,-1}(c_{16})}{C^{*}_{y,-1}(c_{16})} - y , \qquad y=1,2,\ldots,$$

$$m(0) = \frac{-c_{16}}{\log(1-\theta Q^{-1})} ,$$

where $C^{*}_{y,-1}(t)$ is defined under Proposition 4.1 (p) and

$$c_{16} = Q^{-1}(1-\theta Q^{-1})^{-1} ,$$

iff X is a logarithmic (θ); then Y is a logarithmic (θ) v geometric $(1/Q)$.

Proposition 4.4. Suppose (3.9) holds. Then we have:

(a) For some $\lambda > 0$ and some $\theta > 0$

$$m(y) = \frac{S_{y+1}(c_{17})}{S_y(c_{17})} \, , \qquad c_{17} = \theta e^{-\lambda} \, , \quad y = 0, 1, 2, \ldots$$

where $S_y(t)$ is defined under Proposition 4.1 (e)

iff X is a Poisson (θ); then Y is a Poisson $(\theta) v$ Poisson (λ) and

(X, Y) is a bivariate Poisson-Poisson model studied by Leiter and

Hamdan (1973) and Cacoullos and Papageorgiou (1980).

(b) For some $0 < p = 1 - q < 1$ and an integer $n > 0$

$$m(y) = \frac{S_{y+1,n}(c_{18})}{S_{y,n}(c_{18})} \, , \quad y = 0, 1, 2, \ldots,$$

where $S_{y,n}(t)$ is defined under Proposition 4.1 (h) and

$$c_{18} = pe^{-\lambda}(q + pe^{-\lambda})^{-1},$$

iff X is a binomial (n, p); then Y is a binomial $(n, p) v$ Poisson

(λ).

(c) For some $P = Q - 1 > 0$ and an $N > 0$

$$m(y) = \frac{S_{y+1,-N}(-c_{19})}{S_{y,-N}(-c_{19})} \, , \qquad y = 0, 1, 2, \ldots,$$

where $S_{y,-N}(-t)$ is defined under Proposition 4.1 (k) and

$$c_{19} = Pe^{-\lambda}(Q - Pe^{-\lambda})^{-1}.$$

iff X is a negative binomial (N, P) then Y is a negative binomial
$(N, P) v$ Poisson (λ).

(d) For some $0 < \theta < 1$

$$m(y) = \frac{S^*_{y+1}(c_{20})}{S^*_y(c_{20})} \, , \qquad y = 1, 2, \ldots,$$

$$m(0) = \frac{-c_{20}}{\log(1 - \theta e^{-\lambda})} \, ,$$

where $S^*_y(t)$ is defined under Proposition 4.1 (n) and

$$\dot{c}_{20} = \theta e^{-\lambda}(1-\theta e^{-\lambda})^{-1},$$

iff X is a logarithmic (θ); then Y is a logarithmic (θ) v Poisson (λ)

Proposition 4.5 Suppose (3.10) holds. Then we have:

(a) For some $\lambda > 0$

$$m(y) = \frac{C_{y+1,n}(c_{21})}{nC_{y,n}(c_{21})} + \frac{y}{n}, \quad c_{21} = \lambda q^n, \quad y=0,1,2,\ldots,$$

where $C_{y,n}(t)$ is defined under Proposition 4.1 (f)

iff X is a univariate Poisson (λ); then Y is a Poisson (λ) v binomial (n,p) and (X,Y) is a bivariate Poisson-binomial model studied by Cacoullos and Papageorgiou (1980).

(b) For some $0<p'=1-q'<1$ and an integer $n'>0$

$$m(y) = \frac{C_{y+1,n',n}(c_{22})}{nC_{y,n',n}(c_{22})} + \frac{y}{n}, \quad 0\leq y\leq n'n,$$

where $C_{y,n',n}(t)$ is defined under Proposition 4.1 (i) and

$$c_{22} = p'q^n(q'+p'q^n)^{-1},$$

iff X is a binomial (n',p'); then Y is a binomial (n',p') v binomial (n,p).

(c) For some $P=Q-1>0$ and an $N>0$

$$m(y) = \frac{C_{y+1,-N,n}(-c_{23})}{nC_{y,-N,n}(-c_{23})} + \frac{y}{n}, \quad y=0,1,2,\ldots$$

where $C_{y,-N,n}(-t)$ is defined under Proposition 4.1 (ℓ) and

$$c_{23} = Pq^n(Q-Pq^n)^{-1},$$

iff X is a negative binomial (N,P); then Y is a negative binomial (N,P) v binomial (n,p).

(d) For some $0<\theta<1$

$$m(y) = \frac{C^*_{y+1,n}(c_{24})}{nC^*_{y,n}(c_{24})} + \frac{y}{n} , \qquad y=1,2,\ldots,$$

$$m(0) = \frac{-C^*_{1,n}(c_{24})}{n \log(1-\theta q^n)} ,$$

where $C^*_{y,n}(t)$ is defined under Proposition 4.1 (0) and

$$c_{24} = \theta q^n(1-\theta q^n)^{-1},$$

iff X is a univariate logarithmic (θ); then Y is a logarithmic (θ) v binomial (n,p).

5. SOME APPLICATIONS - COROLLARIES OF THEOREM 3.2

Proposition 5.1. Suppose (3.12) holds. Then we have:

(a) For some $\theta>0$

$$m(y) = y-y \frac{\lambda}{\lambda+\theta} , \qquad y=0,1,2,\ldots ,$$

iff X is a univariate Poisson (θ); then Y is also a Poisson ($\theta+\lambda$).

(b) For some $\mu>0$

$$m(y) = y-y \frac{\lambda}{(2\mu)^{1/2}} \frac{H^*_{y-1}(\beta)}{H^*_{y}(\beta)} \qquad y=0,1,2,\ldots$$

where $H^*_y(t)$ are the modified Hermite polynomials, Kemp and Kemp (1965), defined as

$$H^*_y(t) = \sum_{j=0}^{[1/2y]} \frac{y!\,t^{y-2j}}{(y-2j)!\,j!\,2^j} .$$

and $\beta=\lambda(2\mu)^{-1/2}$.

iff X is a doublet Poisson (i.e. X=2Z where Z is a Poisson (μ)); then Y is a Hermite distribution,i.e., a convolution of an ordinary Poisson with a doublet Poisson.

(c) For some $\theta>0$ and some $\mu>0$

$$m(y) = y- \frac{A_{y-1}\{\lambda,\theta,S_j(\mu e^{-\theta})\}}{A_{y}\{\lambda,\theta,S_j(\mu e^{-\theta})\}} , \qquad y=0,1,2,\ldots$$

where

$$A_y\{\lambda,\theta,S_j(t)\} = \sum_{j-0}^{y} \frac{\lambda^{y-j}\theta^j}{(y-j)!j!} S_j(t)$$

and $S_j(t)$ was defined under Proposition 4.1 (e),

iff X is a Neyman (μ,θ); then Y is a "Short" distribution (λ,μ,θ),

Kemp (1967),i.e., a convolution of a Poisson (λ) with a Neyman (μ,θ).

6. SOME STATISTICAL APPLICATIONS OF THE CHARACTERIZATIONS

The preceding results, in addition to their probabilistic interest, can be used in goodness-of-fit tests in a variety of situations.

For illustration, consider the case in which records (X) of accidents and corresponding fatal accidents (Y) are available for a series of periods. Then we may be faced with identifying the distribution of X and Y under the natural assumption that Y given X is binomial. This is the situation described by (3.6). A possible test, within the framework of these characterizations, is to look at the regression function $m(y)$ of X on Y. Thus, if $m(y)$ is linear

$$m(y) = a+by, \qquad y=0,1,\ldots,$$

then (Proposition 4.1 (a),(b),(c)), a regression line with slope $b=1$, shows that X (hence also Y) is a Poisson, a $b<1$ indicates that X (hence also Y) is a binomial and a $b>1$ suggests a negative binomial for X and Y. On the other hand, a line $m(y) = by$ with $b>1$ for $y>1$ and an isolated point at $y=0$ (Proposition 4.1 (d)) indicates a logarithmic X.

Similar remarks can be made concerning the cases of more complicated regression functions, which as a rule, take us away from the simple classical discrete distributions. This, however, is beyond the scope of the present investigation and we shall not pursue it here any further.

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0<p'=1-q'<1 and some integer n>0

$$m(y) = \frac{q'}{q'+p'q} y+n \frac{p'q}{q'+p'q} , \qquad 0 \leq y \leq n.$$

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0<p'=1-q'<1 and some integer n>0

$$m(y) = \frac{q'}{q'+p'q} y+n \frac{p'q}{q'+p'q} , \qquad 0 \leq y \leq n.$$